MATRIX POSITIVITY PRESERVERS IN FIXED DIMENSION. I

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Abstract. A classical theorem proved in 1942 by I.J. Schoenberg describes all real-valued functions that preserve positivity when applied entrywise to positive semidefinite matrices of arbitrary size; such functions are necessarily analytic with non-negative Taylor coefficients. Despite the great deal of interest generated by this theorem, a characterization of functions preserving positivity for matrices of fixed dimension is not known.

In this paper, we provide a complete description of polynomials of degree N that preserve positivity when applied entrywise to matrices of dimension N. This is the key step for us then to obtain negative lower bounds on the coefficients of analytic functions so that these functions preserve positivity in a prescribed dimension. The proof of the main technical inequality is representation theoretic, and employs the theory of Schur polynomials. Interpreted in the context of linear pencils of matrices, our main results provide a closed-form expression for the lowest critical value, revealing at the same time an unexpected spectral discontinuity phenomenon.

Tight linear matrix inequalities for Hadamard powers of matrices and a sharp asymptotic bound for the matrix-cube problem involving Hadamard powers are obtained as applications. Positivity preservers are also naturally interpreted as solutions of a variational inequality involving generalized Rayleigh quotients. This optimization approach leads to a novel description of the simultaneous kernels of Hadamard powers, and a family of stratifications of the cone of positive semidefinite matrices.

Contents

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Date: March 29, 2016.

2010 *Mathematics Subject Classification.* 15A45 (primary); 05E05, 15A12, 15B48, 26C05, 62H99, 65F15 (secondary).

Key words and phrases. positive definite matrix, Hadamard product, Schur polynomial, entrywise function, linear matrix inequality, correlation matrix, spectrahedron, characteristic value.

1. Introduction and main results

Transformations, linear or not, which preserve matrix structures with positivity constraints have been recently studied in at least three distinct frameworks: statistical mechanics and the geometry of polynomials [8, 9, 10]; global optimization algorithms based on the cone of hyperbolic or positive definite polynomials [5, 24, 39]; the statistics of big data, having the correlation matrix of a large number of random variables as the central object [4, 25, 32, 40, 46]. The present article belongs in the latter two categories, although the main result may be of independent algebraic interest.

To describe the contents of this paper, we adopt some terminology. For a set $K \subset \mathbb{C}$ and an integer $N \geq 1$, denote by $\mathcal{P}_N(K)$ the cone of positive semidefinite $N \times N$ matrices with entries in K. A function $f : K \to \mathbb{C}$ naturally acts entrywise on $\mathcal{P}_N(K)$, so that $f[A] := (f(a_{ij}))$ for any $A = (a_{ij}) \in \mathcal{P}_N(K)$. Akin to the theory of positive definite functions, it is natural to seek characterizations of those functions f such that $f[A]$ is positive semidefinite for all $A \in \mathcal{P}_N(K)$. A well-known theorem of Schoenberg [42] states that $f[A]$ is positive semidefinite for all $A \in \mathcal{P}_N([-1, 1])$ of all dimensions $N \geq 1$ if and only if f is absolutely monotonic on $[0, 1]$ (i.e., analytic with non-negative Taylor coefficients). To put Schoenberg's 1942 article in historical perspective, we have to recall that the theory of absolute monotone functions was already established by S. Bernstein [3]. Also, it is worth mentioning that Schoenberg was working around that time on the related and more general question of isometrically embedding positive definite metrics into Hilbert space; see, for instance, [45]. The parallel theory of matrix monotone functions, with $f(A)$ defined by standard functional calculus, owes its main result to Loewner [33] (see also [12]).

Since its publication, Schoenberg's theorem has attracted a great deal of attention. The result has been considered in several different contexts in [1, 6, 7, 11, 28, 41]. See also [2, 15, 16, 29, 36] for recent work.

Schoenberg's observation has a natural application to high-dimensional probability and statistics. Recall that a correlation matrix is the Gram matrix of vectors on a sphere S^{d-1}. In concrete situations, functions are often applied entrywise to correlation matrices, in order to improve their properties, such as better conditioning, or to induce a Markov random field structure; among the recent investigations centered on this technique we note [4, 20, 21, 25, 26, 32, 40, 46]. Whether or not the resulting matrices are positive semidefinite is critical for the validity of these procedures.

According to Schoenberg's theorem, functions preserving positivity when applied entrywise to correlation matrices of all dimensions and bounded rank are non-negative combinations of Gegenbauer polynomials. However, allowing for arbitrary dimension is unnecessarily restrictive, as the state space of the problem is usually known; at the least, its dimension often has an apparent upper bound. Motivated by such practical demands, characterizations of positivity preserving functions have recently been obtained in fixed dimensions, under further constraints that arise in practice; see, for example, [18, 19, 21].

In the case of a fixed dimension N, obtaining characterizations of functions which preserve matrix positivity when applied entrywise remains a difficult open problem, even discouraging in view of the scarcity of known results for low rank and low degree. Using an idea of Loewner, Horn showed in [30] that if a continuous function $f : (0,\infty) \to \mathbb{R}$ satisfies $f[-] : \mathcal{P}_N((0,\infty)) \to \mathcal{P}_N(\mathbb{R})$, then $f \in C^{N-3}((0,\infty))$ and $f^{(k)}(x) \geq 0$ for all $x > 0$ and all $0 \leq k \leq N-3$. Moreover, if it is known that $f \in C^{N-1}((0,\infty))$, then $f^{(k)}(x) \geq 0$ for all $x > 0$ and all $0 \leq k \leq N-1$.

As of today, obtaining an effective characterization of arbitrary functions which preserve matrix positivity in a fixed dimension looks rather inaccessible. The main result of the present article provides the first such characterization for polynomial functions, and illustrates the complexity of the coefficient bounds, even in the simplest of situations.

Theorem 1.1. *Fix $\rho > 0$ and integers $N \geq 1$, $M \geq 0$ and let $f(z) = \sum_{j=0}^{N-1} c_j z^j + c' z^M$ be a polynomial with real coefficients. Also denote by $\overline{D}(0,\rho)$ the closed disc in $\mathbb{C}$ with radius $\rho > 0$ and center the origin. For any vector $\mathbf{d} := (d_0, \dots, d_{N-1})$ with non-zero entries, let*

$$\mathcal{C}(\mathbf{d}) = \mathcal{C}(\mathbf{d}; z^M; N, \rho) := \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{d_j}, \tag{1.1}$$

and let $\mathbf{c} := (c_0, \dots, c_{N-1})$. The following are equivalent.

1. *$f[-]$ preserves positivity on $\mathcal{P}_N(\overline{D}(0,\rho))$.*
2. *The coefficients c_j satisfy either $c_0, \dots, c_{N-1}, c' \geq 0$, or $c_0, \dots, c_{N-1} > 0$ and $c' \geq -\mathcal{C}(\mathbf{c})^{-1}$.*
3. *$f[-]$ preserves positivity on $\mathcal{P}_N^1((0,\rho))$, the set of matrices in $\mathcal{P}_N((0,\rho))$ having rank at most 1.*

Note that the necessity of having $c_0, \dots, c_{N-1} \geq 0$ in part (2) of the theorem follows from Horn's theorem as stated above. The constant $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ provides a threshold for polynomials that preserve positivity on $\mathcal{P}_N$ but not on $\mathcal{P}_{N+1}$. Our theorem thus provides a quantitative version in fixed dimension of Schoenberg's result, as well as of Horn's result. As should be expected, our bound goes asymptotically to the Schoenberg degree-free statement, since $\mathcal{C}(\mathbf{c}; z^M; N, \rho) \to \infty$ as $N \to \infty$. It is remarkable that the proof of Theorem 1.1 is obtained by using Schur polynomials.

Remark 1.2. Notice that if $M < N$ then $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ makes sense and equals c_M^{-1}, if we use for any complex number $z \in \mathbb{C}$ the formulas

$$\binom{z}{n} := \frac{z(z-1)\cdots(z-n+1)}{n!} \quad \text{and} \quad \binom{z}{0} := 1.$$

In effect, Theorem 1.1 says in this case that when f is a polynomial of degree at most $N-1$, the map $f[-]$ preserves positivity on $\mathcal{P}_N(\overline{D}(0,\rho))$ if and only if all coefficients of f are non-negative.

Theorem 1.1 provides a decisive first step towards isolating classes of functions that preserve positivity on $\mathcal{P}_N$ when applied entrywise. Additionally, the result yields a wealth of interesting consequences that initiate the development of an entrywise matrix calculus that leaves invariant the cone $\mathcal{P}_N$, in parallel and in contrast to the much better understood standard functional calculus. The next theorem provides a constructive criterion for preserving positivity, applicable to all analytic functions. In the theorem and thereafter, the $N \times N$ matrix with all entries equal to 1 is denoted by $\mathbf{1}_{N\times N}$, and $A^{\circ k} := (a_{ij}^k)$ denotes the kth Hadamard (or entrywise, or Schur) power of A.

Theorem 1.3. *Fix $\rho > 0$ and an integer $N \geq 1$. Let $\mathbf{c} := (c_0, \ldots, c_{N-1}) \in (0,\infty)^N$, and suppose $g(z) := \sum_{M=N}^\infty c_M z^M$ is analytic on $D(0,\rho)$ and continuous on $\overline{D(0,\rho)}$, with real coefficients. Then*

$$t(c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}) - g[A] \in \mathcal{P}_N(\mathbb{C}) \tag{1.2}$$

for all $A \in \mathcal{P}_N(\overline{D}(0,\rho))$ and all

$$t \geq \sum_{M \geq N : c_M > 0} c_M \mathcal{C}(\mathbf{c}; z^M; N, \rho).$$

Moreover, this series is convergent, being bounded above by

$$\frac{g_2^{(2N-2)}(\sqrt{\rho})}{2^{N-1}(N-1)!^2} \sum_{j=0}^{N-1} \binom{N-1}{j}^2 \frac{\rho^{N-j-1}}{c_j}, \tag{1.3}$$

where $g_2(z) := g_+(z^2)$ and

$$g_+(z) := \sum_{M \geq N : c_M > 0} c_M z^M.$$

Note that Theorem 1.1 concerns the special case of Theorem 1.3 with $g(z) = c_M z^M$.

Theorem 1.3 provides a sufficient condition for a large class of functions to preserve positive semidefiniteness in fixed dimension. The loosening of the tight thresholds for the individual coefficients is compensated in this case by the closed form of the bound for the lowest eigenvalue of the respective matrix pencil.

Next we describe some consequences of our main results. For $A \in \mathcal{P}_N(K)$ and f as in Theorem 1.1 with $M \geq N$, note that

$$f[A] = c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)} + c_M A^{\circ M},$$

where $c_M = c'$. Understanding when $f[A]$ is positive semidefinite is thus equivalent to controlling the spectrum of linear combinations of Hadamard powers of A, by obtaining linear inequalities of the form

$$c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)} + c_M A^{\circ M} \geq 0,$$

where the order is the Loewner ordering, given by the cone $\mathcal{P}_N(\mathbb{C})$. A direct application of our main theorem provides a sharp bound for controlling the Hadamard powers of positive semidefinite matrices.

Corollary 1.4. *Fix $\rho > 0$, integers $M \geq N \geq 1$, and scalars $c_0, \ldots, c_{N-1} > 0$. Then*

$$A^{\circ M} \leq \mathcal{C}(\mathbf{c}; z^M; N, \rho) \cdot \left(c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}\right) \tag{1.4}$$

for all $A \in \mathcal{P}_N(\overline{D}(0,\rho))$. Moreover, the constant $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ is sharp.

As an immediate consequence of Lemma 2.4 below, notice that the right-hand side of Equation (1.4) cannot be replaced by a sum of fewer than N Hadamard powers of A. Corollary 1.4 thus yields a sharp bound for controlling the Hadamard power $A^{\circ M}$ with the smallest number of powers of lower order.

In a different direction, our main result naturally fits into the fast developing area of spectrahedra [5, 44] and the matrix cube problem [37]. The latter, a key technical ingredient in modern optimization theory, continues to attract the attention of various groups of researchers, mostly in applied mathematics. Recall that given real symmetric $N \times N$ matrices $A_0, \ldots, A_{M+1}$, where $M \geq 0$, the corresponding matrix cubes are

$$\mathcal{U}[\eta] := \left\{ A_0 + \sum_{m=1}^{M+1} u_m A_m : u_m \in [-\eta, \eta] \right\} \qquad (\eta > 0). \tag{1.5}$$

The matrix cube problem consists of determining whether $\mathcal{U}[\eta] \subset \mathcal{P}_N$, and finding the largest η for which this is the case. As another consequence of our main result, we obtain an asymptotically sharp bound for the matrix cube problem when the matrices A_m are Hadamard powers.

Corollary 1.5. *Fix $\rho > 0$ and integers $M \geq 0$, $N \geq 1$. Given a matrix $A \in \mathcal{P}_N(\overline{D}(0,\rho))$, let*

$$A_0 := c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}$$

and

$$A_m := A^{\circ(N-1+m)} \qquad \text{for } 1 \leq m \leq M+1,$$

where the coefficients $c_0, \ldots, c_{N-1} > 0$. Then

$$\eta \leq \left(\sum_{m=0}^{M} \mathcal{C}(\mathbf{c}; z^{N+m}; N, \rho) \right)^{-1} \Rightarrow \mathcal{U}[\eta] \subset \mathcal{P}_N(\mathbb{C}) \tag{1.6}$$

$$\Rightarrow \eta \leq \mathcal{C}(\mathbf{c}; z^{N+M}; N, \rho)^{-1}. \tag{1.7}$$

The upper and lower bounds for η are asymptotically equal as $N \to \infty$, i.e.,

$$\lim_{N\to\infty} \mathcal{C}(\mathbf{c}; z^{N+M}; N, \rho)^{-1} \sum_{m=0}^{M} \mathcal{C}(\mathbf{c}; z^{N+m}; N, \rho) = 1. \tag{1.8}$$

See the end of Section 3.3 for the proof of this result.

Finally, understanding which polynomials preserve positivity on $\mathcal{P}_N(K)$ can be reformulated as an extremal problem involving generalized Rayleigh quotients of Hadamard powers.

Theorem 1.6. *Fix $\rho > 0$, integers $M \geq N \geq 1$, and scalars $c_0, \ldots, c_{N-1} > 0$. Then*

$$\inf_{\mathbf{v} \in \mathcal{K}(A)^\perp \setminus \{0\}} \frac{\mathbf{v}^* \left(\sum_{j=0}^{N-1} c_j A^{\circ j} \right) \mathbf{v}}{\mathbf{v}^* A^{\circ M} \mathbf{v}} \geq \mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$$

for all non-zero $A \in \mathcal{P}_N(\overline{D}(0, \rho))$, where

$$\mathcal{K}(A) := \ker(c_0 \mathbf{1}_{N \times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}).$$

The lower bound $\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$ is sharp, and may be obtained by considering only the set of rank-one matrices $\mathcal{P}_N^1((0, \rho))$.

We note the surprising fact that the left-hand side of the inequality above is *not* continuous in the variable A; see Remark 4.7. Motivated by this approach, we obtain, in Theorem 5.7 below, a description of the kernel $\mathcal{K}(A)$ for a given matrix $A \in \mathcal{P}_N(\mathbb{C})$. As we show, this kernel coincides with the simultaneous kernels $\cap_{n \geq 0} \ker A^{\circ n}$ of the Hadamard powers of A, and leads to a hitherto unexplored stratification of the space $\mathcal{P}_N(\mathbb{C})$.

The rest of this paper is organized as follows. We review background material in Section 2. The main result of the paper is proved in Section 3, along with many intermediate results on Schur polynomials that may be interesting in their own right. We also show in Section 3 how our main result can naturally be extended to general polynomials, and to analytic functions. Section 4 deals with the reformulation of our main theorem as a variational problem, and provides a closed-form expression for the extreme critical value of a single positive semidefinite matrix. Beginning with a novel block-matrix decomposition of such matrices into rank-one components, Section 5 provides a description of the simultaneous kernels of Hadamard powers of a positive semidefinite matrix. The last section contains a unified presentation of a dozen known computations in closed form, of extreme critical values for matrix pencils, all relevant to our present work.

Acknowledgments. The authors are grateful to Bala Rajaratnam for sharing his enthusiasm and ideas that attracted the four of us to these topics. We would like to thank the American Institute of Mathematics (AIM) for hosting the workshop "Positivity, graphical models, and modeling of complex multivariate dependencies" in October 2014, where this project was initiated. We thank Yuan Xu for very stimulating conversations at the AIM workshop, which were helpful in embarking upon this project, and Shmuel Friedland, for valuable insights into generalized Rayleigh quotients. We are indebted to the referee for a careful examination of the contents of the manuscript and constructive criticism on its presentation.

2. Background and notation

Given a subset $K \subset \mathbb{C}$ and integers $1 \leq k \leq N$, let $\mathcal{P}_N^k(K)$ denote the set of positive semidefinite $N \times N$ matrices with entries in K and with rank at most k, and let $\mathcal{P}_N(K) := \mathcal{P}_N^N(K)$. Given a matrix A, let $A^{\circ k}$ denote the matrix obtained from A by taking the kth power of each entry; in particular, if A is an $N \times N$ matrix then $A^{\circ 0} = \mathbf{1}_{N\times N}$, the $N \times N$ matrix with each entry equal to 1.

Recall that the *Gegenbauer* or *ultraspherical* polynomials $C_n^{(\lambda)}(x)$ satisfy

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n \qquad (\lambda > 0),$$

while for the *Chebyshev polynomials of the first kind* $C_n^{(0)}(x)$ we have

$$(1 - xt)(1 - 2xt + t^2)^{-1} = \sum_{n=0}^{\infty} C_n^{(0)}(x) t^n.$$

We begin by recalling Schoenberg's original statement, which classifies positive definite functions on a sphere S^{d-1} of fixed dimension.

Theorem 2.1 (Schoenberg, [42, Theorems 1 and 2])**.** *Fix an integer $d \geq 2$ and a continuous function $f : [-1, 1] \to \mathbb{R}$.*

(1) *The function $f(\cos \cdot)$ is positive definite on the unit sphere S^{d-1} if and only if f can be written as a non-negative linear combination of the Gegenbauer or Chebyshev polynomials $C_n^{(\lambda)}$, where $\lambda = (d-2)/2$:*

$$f(x) = \sum_{n\geq 0} a_n C_n^{(\lambda)}(x) \qquad (a_n \geq 0).$$

(2) *The entrywise function $f[-] : \mathcal{P}_N([-1, 1]) \to \mathcal{P}_N(\mathbb{R})$ for all $N \geq 1$ if and only if f is analytic on $[-1, 1]$ and absolutely monotonic on $[0, 1]$, i.e., f has a Taylor series with non-negative coefficients convergent on the closed unit disc $\overline{D}(0, 1)$.*

For more on absolutely monotonic functions, see the work [3] of Bernstein. Rudin [41] proved part (2) of the above result without the continuity assumption, in addition to several other characterizations of such a function f. His work was a part of the broader context of studying functions acting on Fourier–Stieltjes transforms in locally compact groups, as explored in joint works with Kahane, Helson, and Katznelson in [23, 31].

The work of Schoenberg has subsequently been extended along several directions; see, for example, [1, 6, 7, 11, 28, 29, 45]. However, the solution to the original problem in fixed dimension remains elusive when $N > 2$.

An interesting necessary condition for a continuous function $f : (0, \infty) \to \mathbb{R}$ to preserve positivity in fixed dimension has been provided by Horn [30]. This result was recently extended in [18] to apply in the case of low-rank matrices with entries in $(0, \rho)$ for some $\rho > 0$, and without the continuity assumption; on the last point, see also Hiai's work [29].

Theorem 2.2 (Horn [30], Guillot–Khare–Rajaratnam [18])**.** *Suppose $f : I \to \mathbb{R}$, where $I := (0, \rho)$ and $0 < \rho \leq \infty$. Fix an integer $N \geq 2$ and suppose that $f[A] \in \mathcal{P}_N(\mathbb{R})$ for any $A \in \mathcal{P}_N^2(I)$ of the form $A = a\mathbf{1}_{N\times N} + \mathbf{u}\mathbf{u}^T$, where $a \in (0, \rho)$ and $\mathbf{u} \in [0, \sqrt{\rho - a})^N$. Then $f \in C^{N-3}(I)$, with*

$$f^{(k)}(x) \geq 0 \qquad \forall x \in I,\ 0 \leq k \leq N-3,$$

and $f^{(N-3)}$ is a convex non-decreasing function on I. If, further, $f \in C^{N-1}(I)$, then $f^{(k)}(x) \geq 0$ for all $x \in I$ and $0 \leq k \leq N-1$.

Note that all real power functions preserve positivity on $\mathcal{P}_N^1((0,\rho))$, yet such functions need not have even a single positive derivative on $(0,\rho)$. However, as shown in Theorem 2.2, working with a small one-parameter extension of $\mathcal{P}_N^1((0,\rho))$ guarantees that $f^{(k)}$ is non-negative on $(0,\rho)$ for $0 \leq k \leq N-3$.

Remark 2.3. Theorem 2.2 is sharp in the sense that there exist functions $f : (0,\rho) \to \mathbb{R}$ which preserve positivity on $\mathcal{P}_N((0,\rho))$, but not on $\mathcal{P}_{N+1}((0,\rho))$. For example, $f(x) = x^\alpha$, where $\alpha \in (N-2, N-1)$ is an example of such a function; see [13, 17, 29] for more details. Thus the bound on the number of non-negative derivatives in Theorem 2.2 is sharp.

In light of Remark 2.3, we focus henceforth on analytic functions which preserve $\mathcal{P}_N(K)$ for fixed N when applied entrywise. Note that any analytic function on $D(0,\rho)$ that maps $(0,\rho)$ to $\mathbb{R}$ necessarily has real Taylor coefficients.

Recall by Theorem 2.2 that if $f[-] : \mathcal{P}_N^2((0,\infty)) \to \mathcal{P}_N(\mathbb{R})$ and $f \in C^{(N-1)}((0,\infty))$ then $f^{(k)}$ is non-negative on $(0,\infty)$ for $0 \leq k \leq N-1$. The next lemma shows that if f is assumed to be analytic, then it suffices to work with $\mathcal{P}_N^1$ instead of $\mathcal{P}_N^2$ in order to arrive at the same conclusion.

Lemma 2.4. *Let $f : D(0,\rho) \to \mathbb{R}$ be an analytic function, where $0 < \rho \leq \infty$, so that $f(x) = \sum_{n \geq 0} c_n x^n$ on $D(0,\rho)$. If $f[-] : \mathcal{P}_N^1((0,\rho)) \to \mathcal{P}_N(\mathbb{R})$ for some integer $N \geq 1$, then the first N non-zero Taylor coefficients c_j are strictly positive.*

In particular, if f is a sum of at most N monomials, then under the hypotheses of Lemma 2.4, all coefficients of f are non-negative.

Proof. Suppose that the first m non-zero coefficients are $c_{n_1}, \dots, c_{n_m}$, where $m \leq N$. Fix $\mathbf{u} := (u_1, \dots, u_m)^T \in (0, \sqrt{\rho})^m$ with distinct entries $u_1, \dots, u_m$, and note the matrix $(u_j^{n_k})_{j,k=1}^m$ is non-singular [14, Chapter XIII, §8, Example 1]. Hence $\{\mathbf{u}^{\circ n_1}, \dots, \mathbf{u}^{\circ n_m}\}$ is linearly independent and there exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^m$ such that $(\mathbf{u}^{\circ n_j})^T \mathbf{v}_k = \delta_{j,k}$ for all $1 \leq j, k \leq m$. Since f preserves positivity on $\mathcal{P}_N^1((0,\rho))$, it follows that

$$0 \leq \epsilon^{-n_k} \mathbf{v}_k^T f[\epsilon \mathbf{u}\mathbf{u}^T] \mathbf{v}_k = c_{n_k} + \sum_{j > n_m} c_j (\mathbf{u}^{\circ j} \mathbf{v}_k)^2 \epsilon^{j - n_k} \to c_{n_k}$$

as $\epsilon \to 0^+$, for $1 \leq k \leq m$. The result follows. $\square$

The above discussion naturally raises various questions.

(1) Can one find necessary and sufficient conditions on a restricted class of functions, such as polynomials, to ensure that positivity is preserved in fixed dimension?
(2) Note that the power functions in Remark 2.3, $f(x) = x^\alpha$ for $\alpha \in (N-2, N-1)$, are not analytic. Does there exist a function f analytic on an open subset $U \subset \mathbb{C}$ which preserves positivity on $\mathcal{P}_N(U)$, but not on $\mathcal{P}_{N+1}(U)$?

Our main result provides positive answers to both of these questions.

3. Schur polynomials and Hadamard powers

The goal of this section is to prove Theorem 1.1. The proof relies on a careful analysis of the polynomial

$$p(t) := \det(t(c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}) - A^{\circ M}),$$

where $A \in \mathcal{P}_N^1(\overline{D}(0,\rho))$. More specifically, we study algebraic properties of the polynomial $p(t)$, and show how an explicit factorization can be obtained by exploiting the theory of symmetric polynomials.

3.1. **Determinantal identities for Hadamard powers.** We begin with some technical preliminaries involving Schur polynomials.

As the results in this subsection may be of independent interest to specialists in symmetric functions and algebraic combinatorics, we state them over an arbitrary field $\mathbb{F}$.

Given a partition, i.e., a non-increasing N-tuple of non-negative integers $\mathbf{n} = (n_N \geq \cdots \geq n_1)$, the corresponding *Schur polynomial* $s_{\mathbf{n}}(x_1, \dots, x_N)$ over a field $\mathbb{F}$ with at least N elements is defined to be the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(x_1, \dots, x_N) := \frac{\det(x_i^{n_j+N-j})}{\det(x_i^{N-j})} \tag{3.1}$$

for pairwise distinct $x_i \in \mathbb{F}$. Note that the denominator is precisely the Vandermonde determinant $\Delta_N(x_1, \dots, x_N) := \det(x_i^{N-j}) = \prod_{1\leq i<j\leq N}(x_i - x_j)$; it follows from this that

$$s_{\mathbf{n}}(1, \dots, z^{N-1}) = \prod_{1\leq i<j\leq N} \frac{z^{n_j+j} - z^{n_i+i}}{z^j - z^i}, \quad s_{\mathbf{n}}(1, \dots, 1) = \prod_{1\leq i<j\leq N} \frac{n_j - n_i + j - i}{j - i}. \tag{3.2}$$

The last equation can also be deduced from Weyl Character Formula in type A; see, for example, [34, Chapter I.3, Example 1]. For more details about Schur polynomials and the theory of symmetric functions, see [34]. In particular, note that Schur polynomials have non-negative integer coefficients, by [34, Chapter I, Equation (5.12)].

Proposition 3.1. *Let $A := \mathbf{u}\mathbf{v}^T$, where $\mathbf{u} = (u_1, \dots, u_N)^T$ and $\mathbf{v} := (v_1, \dots, v_N)^T \in \mathbb{F}^N$ for $N \geq 1$. Given m-tuples of non-negative integers $\mathbf{n} = (n_m > n_{m-1} > \cdots > n_1)$ and scalars $(c_{n_1}, \dots, c_{n_m}) \in \mathbb{F}^m$, the following determinantal identity holds:*

$$\det \sum_{j=1}^m c_{n_j} A^{\circ n_j} = \Delta_N(\mathbf{u})\Delta_N(\mathbf{v}) \sum_{\mathbf{n}'\subset\mathbf{n},\ |\mathbf{n}'|=N} s_{\lambda(\mathbf{n}')}(\mathbf{u}) s_{\lambda(\mathbf{n}')}(\mathbf{v}) \prod_{k=1}^N c_{n'_k}. \tag{3.3}$$

Here, $\lambda(\mathbf{n}') := (n'_N - N + 1 \geq n'_{N-1} - N + 2 \geq \cdots \geq n'_1)$ is obtained by subtracting the staircase partition $(N-1, \dots, 0)$ from $\mathbf{n}' := (n'_N > \cdots > n'_1)$, and the sum is over all subsets $\mathbf{n}'$ of cardinality N. In particular, if $m < N$ then the determinant is zero.

Proof. If there are $m < N$ summands then the matrix in question has rank at most $m < N$, so it is singular; henceforth we suppose $m \geq N$. Note first that if $\mathbf{c} := (c_{n_1}, \dots, c_{n_m})$ and

$$X(\mathbf{u}, \mathbf{n}, \mathbf{c}) := (\sqrt{c_{n_k}} u_j^{n_k})_{1\leq j\leq N, 1\leq k\leq m}$$

where we work over an algebraic closure of $\mathbb{F}$, then

$$\sum_{j=1}^{m} c_{n_j} A^{\circ n_j} = X(\mathbf{u}, \mathbf{n}, \mathbf{c}) X(\mathbf{v}, \mathbf{n}, \mathbf{c})^T. \tag{3.4}$$

Next, let $\mathbf{c}|_{\mathbf{n}'} := (c_{n'_1}, \ldots, c_{n'_N})$ and note that, by the Cauchy–Binet formula applied to (3.4),

$$\begin{aligned} \det \sum_{j=1}^{m} c_{n_j} A^{\circ n_j} &= \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} \det\big(X(\mathbf{u}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'}) X(\mathbf{v}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'})^T\big) \\ &= \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} \det X(\mathbf{u}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'}) \det X(\mathbf{v}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'}) \\ &= \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} \det(u_j^{n'_k}) \det(v_j^{n'_k}) \prod_{k=1}^{N} c_{n'_k}. \end{aligned}$$

Each of the last two determinants is precisely the product of the appropriate Vandermonde determinant times the Schur polynomial corresponding to $\lambda(\mathbf{n}')$; see Equation (3.1). This observation completes the proof. $\square$

We now use Proposition 3.1 to obtain an explicit factorization of the determinant of $p[\mathbf{u}\mathbf{v}^T]$ for a large class of polynomials p.

Theorem 3.2. *Let $R \geq 0$ and $M \geq N \geq 1$ be integers, let $c_0, \ldots, c_{N-1} \in \mathbb{F}^\times$ be non-zero scalars, and let the polynomial*

$$p_t(x) := t(c_0 x^R + \cdots + c_{N-1} x^{R+N-1}) - x^{R+M},$$

where t is a variable. Let the hook partition $\mu(M, N, j) := (M-N+1, 1, \ldots, 1, 0, \ldots, 0)$, with $N-j-1$ entries after the first equal to 1 and the remaining j entries equal to 0. The following identity holds for all $\mathbf{u} = (u_1, \ldots, u_N)$ and $\mathbf{v} := (v_1, \ldots, v_N) \in \mathbb{F}^N$:

$$\det p_t[\mathbf{u}\mathbf{v}^T] = t^{N-1} \Delta_N(\mathbf{u}) \Delta_N(\mathbf{v}) \prod_{j=1}^{N} c_{j-1} u_j^R v_j^R \Big(t - \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\mathbf{u}) s_{\mu(M,N,j)}(\mathbf{v})}{c_j} \Big). \tag{3.5}$$

Proof. Let $A = \mathbf{u}\mathbf{v}^T$ and note first that $\det(A^{\circ R} \circ B) = \prod_{j=1}^{N} u_j^R v_j^R \cdot \det B$ for any $N \times N$ matrix B, so it suffices to prove the result when $R = 0$, which we assume from now on.

Recall the Laplace formula: if B and C are $N \times N$ matrices, then

$$\det(B + C) = \sum_{\mathbf{n} \subset \{1, \ldots, N\}} \det M_{\mathbf{n}}(B; C), \tag{3.6}$$

where $M_{\mathbf{n}}(B; C)$ is the matrix formed by replacing the rows of B labelled by elements of $\mathbf{n}$ with the corresponding rows of C. In particular, if $B = \sum_{j=0}^{N-1} c_j A^{\circ j}$ then

$$\det p_t[A] = \det(tB - A^{\circ M}) = t^N \det B - t^{N-1} \sum_{j=1}^{N} \det M_{\{j\}}(B; A^{\circ M}), \tag{3.7}$$

since the determinants in the remaining terms contain two rows of the rank-one matrix $A^{\circ M}$. By Proposition 3.1 applied with $n_j = j-1$, we obtain

$$\det B = \Delta_N(\mathbf{u})\Delta_N(\mathbf{v})c_0\cdots c_{N-1}.$$

To compute the coefficient of t^{N-1}, note that taking $t=1$ in Equation (3.7) gives that

$$\sum_{j=1}^{N} \det M_{\{j\}}(B; A^{\circ M}) = \det B - \det p_1[A].$$

Moreover, $\det p_1[A]$ can be computed using Proposition 3.1 with $m = N+1$ and $c_{n_{N+1}} = -1$:

$$\det p_1[A] = \det B - \Delta_N(\mathbf{u})\Delta_N(\mathbf{v})c_0\cdots c_{N-1}\sum_{j=0}^{N-1}\frac{s_{\mu(M,N,j)}(\mathbf{u})s_{\mu(M,N,j)}(\mathbf{v})}{c_j},$$

since $\mu(M,N,j) = \lambda\big((M, N-1, N-2, \dots, j+1, \widehat{j}, j-1, \dots, 0)\big)$ for $0 \leq j \leq N-1$. The identity (3.5) now follows. $\square$

Remark 3.3. The connection between Theorem 3.2 and the constant $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ in Theorem 1.1 stems from the fact that

$$s_{\mu(M,N,j)}(1,\dots,1) = \binom{M}{j}\binom{M-j-1}{N-j-1} \tag{3.8}$$

for $0 \leq j \leq N-1$. This is a straightforward application of Equation (3.2); alternatively, it follows by applying Stanley's hook-content formula [43, Theorem 15.3] to the hook Schur function $\mu(M,N,j)$. We also mention a third proof using the dual Jacobi–Trudi (Von Nägelsbach–Kostka) identity [34, Chapter I, Equation (3.5)], which rewrites Schur polynomials in terms of elementary symmetric polynomials. The proof goes as follows: note that the dual partition of $\mu(M,N,j)$ is, up to attaching zeros at the end, the $(M+1)$-tuple $\mu'(M,N,j) := (N-j, 1, \dots, 1)$. Therefore $s_{\mu(M,N,j)}(1,\dots,1)$ equals the determinant of a 2×2 block triangular matrix, one of whose block diagonal submatrices is unipotent, and the determinant of the other is computed inductively to yield (3.8).

Corollary 3.4. *In the setting of Theorem 3.2, if $M = N$, then*

$$\det p_t[A] = t^{N-1}\Delta_N(\mathbf{u})\Delta_N(\mathbf{v})\prod_{j=1}^{N} c_{j-1}u_j^R v_j^R\Big(t - \sum_{j=0}^{N-1}\frac{e_{N-j}(\mathbf{u})e_{N-j}(\mathbf{v})}{c_j}\Big), \tag{3.9}$$

where $e_j(\mathbf{u}) := \sum_{1\leq n_1<\cdots<n_j\leq N} u_{n_1}\dots u_{n_j}$ is the elementary symmetric polynomial in N variables of degree j.

Proof. This follows immediately from the observation that $s_{\mu(0,N,j)}$ is equal to e_{N-j}; see [34, Chapter I, Equation (3.9)]. $\square$

We conclude this part with an identity, which shows how the Schur polynomials $s_{\mu(M,N,j)}$ in Theorem 3.2 can be used to express the Hadamard power $A^{\circ M}$ as a combination of lower Hadamard powers.

Lemma 3.5. *Fix integers $M \geq N \geq 1$, and let the $N \times N$ matrix A have entries in $\mathbb{F}$. Denote the rows of A by $\mathbf{a}_1, \ldots, \mathbf{a}_N$. Then*

$$A^{\circ M} = \sum_{j=0}^{N-1} D_{M,j}(A) A^{\circ j}, \tag{3.10}$$

where $D_{M,j}(A)$ is the diagonal matrix

$$(-1)^{N-j-1} \operatorname{diag}\big(s_{\mu(M,N,j)}(\mathbf{a}_1), \ldots, s_{\mu(M,N,j)}(\mathbf{a}_N)\big),$$

and $s_{\mu(M,N,j)}$ is as in Theorem 3.2.

Proof. Let $\mathbf{v} := (v_1, \ldots, v_N)$, where $v_1, \ldots, v_N$ are pairwise distinct and transcendental over $\mathbb{F}$; we now work in the field $\mathbb{F}(v_1, \ldots, v_n)$. If V is the Vandermonde matrix (v_i^{j-1}) then, by Cramer's rule, the solution to the equation

$$V\mathbf{u} = (\mathbf{v}^{\circ M})^T \qquad \Longleftrightarrow \qquad \mathbf{v}^{\circ M} = \sum_{j=0}^{N-1} u_{j+1} \mathbf{v}^{\circ j}$$

is given by setting u_i to equal the Schur polynomial $s_{\mu(M,N,i-1)}(\mathbf{v})$ times $(-1)^{N-i}$, to account for transposing the ith column to the final place. The result now follows from this identity, applied by specializing $\mathbf{v}$ to each row of A. □

3.2. **Proof of the main theorem.** Using the technical results on Schur polynomials established above, we can now prove Theorem 1.1.

Proof of Theorem 1.1. We begin by proving the result when $0 \leq M \leq N-1$. In this case, it follows immediately from Lemma 2.4 that the theorem holds, since $\mathcal{C}(\mathbf{c}; z^M; N, \rho) = c_M^{-1}$ by Remark 1.2.

Now assume $M \geq N$. To be consistent with the statement of the theorem, where the floating coefficient is denoted by c', for $M \geq N$ we adopt the unifying notation $c_M = c'$. Clearly (1) implies (3). We will now show that (3) implies (2), and (2) implies (1).

(**3**) $\implies$ (**2**). Suppose that $f[-]$ preserves positivity on $\mathcal{P}_N^1((0,\rho))$. By Lemma 2.4, the first N non-zero coefficients of f are positive, so we suppose $c_M < 0$ and prove that $c_M \geq -\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1}$. Define $p_t(x)$ as in Theorem 3.2, with $R = 0$, and set $t := |c_M|^{-1}$, so that $|c_M|^{-1} f[-] = p_t[-]$ preserves positivity on rank-one matrices $A = \mathbf{u}\mathbf{u}^T$ with $\mathbf{u} \in (0, \sqrt{\rho})^n$. Then, by Equation (3.5),

$$0 \leq \det p_t[\mathbf{u}\mathbf{u}^T] = t^{N-1} \Delta_N(\mathbf{u})^2 c_0 \cdots c_{N-1} \Big(t - \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\mathbf{u})^2}{c_j} \Big). \tag{3.11}$$

Now set $u_k := \sqrt{\rho}(1 - t'\epsilon_k)$, with $\epsilon_k \in (0,1)$ pairwise distinct and $t' \in (0,1)$, so that $\Delta_N(\mathbf{u}) \neq 0$. Taking the limit as $t' \to 0$, since the final term in Equation (3.11) must be non-negative, we conclude that

$$\begin{aligned} t = |c_M|^{-1} \geq \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\sqrt{\rho}, \ldots, \sqrt{\rho})^2}{c_j} &= \sum_{j=0}^{N-1} s_{\mu(M,N,j)}(1, \ldots, 1)^2 \frac{\rho^{M-j}}{c_j} \\ &= \mathcal{C}(\mathbf{c}; z^M; N, \rho), \end{aligned} \tag{3.12}$$

as claimed.

(**2**) $\implies$ (**1**). Suppose (2) holds. Note that (1) follows if $c_j \geq 0$ for all j, by the Schur product theorem, so we assume that $c_0, \ldots, c_{N-1} > 0$ and $-\mathcal{C}(\mathbf{c}; z^M; N, \rho)^{-1} \leq c_M < 0$.

We first show that $f[-]$ preserves positivity on $\mathcal{P}_N^1(\overline{D}(0,\rho))$. For $1 \leq m \leq N$, let

$$C_m := \sum_{j=0}^{m-1} s_{\mu(M-N+m,m,j)}(1,\ldots,1)^2 \frac{\rho^{m+M-N-j}}{c_{N-m+j}} = \mathcal{C}(\mathbf{c}_m; z^{M-N+m}; m, \rho), \tag{3.13}$$

where $\mathbf{c}_m := (c_{N-m}, \ldots, c_{N-1})$. Note in this case that C_N is precisely $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ as defined in Equation (1.1) (and N, M are fixed). It follows from Theorem 3.2 that $C_1 = \rho^{M-N+1}/c_{N-1}$ and, for $1 \leq m \leq N-1$,

$$\begin{aligned}
&C_{m+1} - C_m \\
&\geq \sum_{j=0}^{m-1} \big(s_{\mu(M-N+m+1,m+1,j+1)}(1,\ldots,1)^2 - s_{\mu(M-N+m+1,m,j)}(1,\ldots,1)^2 \big) \frac{\rho^{m+M-N-j}}{c_{N-m+j}} \\
&> 0,
\end{aligned}$$

since

$$\begin{aligned}
\frac{s_{\mu(M-N+m+1,m+1,j+1)}(1,\ldots,1)}{s_{\mu(M-N+m,m,j)}(1,\ldots,1)} &= \binom{m+1+M-N}{j+1} \binom{m+M-N}{j}^{-1} \\
&= \frac{m+1+M-N}{j+1} > 1.
\end{aligned}$$

Thus $0 < C_1 < C_2 < \cdots < C_N$.

Next, we claim that for $1 \leq m \leq N$ and all $A = \mathbf{u}\mathbf{u}^* \in \mathcal{P}_N^1(\overline{D}(0,\rho))$, every principal $m \times m$ submatrix of the matrix

$$L := C_m(c_{N-m}\mathbf{1}_{N\times N} + c_{N-m+1}A + \cdots + c_{N-1}A^{\circ(m-1)}) - A^{\circ(m+M-N)} \tag{3.14}$$

is positive semidefinite; for $m = N$, this gives immediately the rank-one case of (1).

The proof of the claim is by induction. The case $m = 1$ is immediate, since a general diagonal entry of L for $m = 1$ equals $C_1 c_{N-1} - a_{jj}^{M-N+1} = \rho^{M-N+1} - a_{jj}^{M-N+1} \geq 0$. Now suppose the result holds for $m - 1 \geq 1$. In the remainder of this proof, we adopt the following notation: given a non-empty set $\mathbf{n} \subset \{1, \ldots, N\}$ and an $N \times N$ matrix B, denote by $B_{\mathbf{n}}$ the principal submatrix of B consisting of those rows and columns labelled by elements of J; we adopt a similar convention for the subvector $\mathbf{u}_{\mathbf{n}}$ of a vector $\mathbf{u}$. If $\mathbf{n} \subset \{1, \ldots, N\}$ has cardinality m then, by Theorem 3.2 with $\mathbf{v} = \overline{\mathbf{u}_{\mathbf{n}}}$,

$$\det L_{\mathbf{n}} = C_m^{m-1} |\Delta_m(\mathbf{u}_{\mathbf{n}})|^2 \prod_{j=1}^m c_{N-j} \Big(C_m - \sum_{j=0}^{m-1} \frac{|s_{\mu(M-N+m,m,j)}(\mathbf{u}_{\mathbf{n}})|^2}{c_{N-m+j}} \Big).$$

Using the triangle inequality in $\mathbb{C}$ and the fact that the coefficients of any Schur polynomial are non-negative, it follows immediately that

$$\begin{aligned}
|s_{\mu(M-N+m,m,j)}(\mathbf{u}_{\mathbf{n}})|^2 &\leq s_{\mu(M-N+m,m,j)}(\sqrt{\rho}, \ldots, \sqrt{\rho})^2 \\
&= s_{\mu(M-N+m,m,j)}(1,\ldots,1)^2 \rho^{m+M-N-j},
\end{aligned} \tag{3.15}$$

and so, by the choice of C_m, the determinant $\det L_{\mathbf{n}} \geq 0$ for any set $\mathbf{n} \subset \{1, \dots, N\}$ of cardinality m. Additionally, for each non-empty subset $\mathbf{n}$ of cardinality $k < m$,

$$\begin{aligned} L_{\mathbf{n}} &\geq C_m(c_{N-k} A_{\mathbf{n}}^{\circ(m-k)} + \cdots + c_{N-1} A_{\mathbf{n}}^{\circ(m-1)}) - A_{\mathbf{n}}^{\circ(m+M-N)} \\ &\geq A_{\mathbf{n}}^{\circ(m-k)} \circ \big(C_k(c_{N-k}\mathbf{1}_{k\times k} + \cdots + c_{N-1} A_{\mathbf{n}}^{\circ(k-1)}) - A_{\mathbf{n}}^{\circ(k+M-N)}\big), \end{aligned}$$

since $C_m > C_k$. It follows by the Schur product theorem and the induction hypothesis that $\det L_{\mathbf{n}} \geq 0$ for any non-empty subset $\mathbf{n}$ of cardinality $k \leq m$. Hence all principal $m \times m$ submatrices of L are positive semidefinite, which concludes the proof of the claim by induction. In particular, $f[-]$ preserves positivity on $\mathcal{P}_N^1(\overline{D}(0,\rho))$.

We now prove that (1) holds for matrices of any rank; again we proceed by induction. For $N = 1$, the result holds because $\mathcal{P}_1(\overline{D}(0,\rho)) = \mathcal{P}_1^1(\overline{D}(0,\rho))$. Now suppose (1) holds for $N - 1 \geq 1$, and let

$$p_t[B; M, \mathbf{d}] := t(d_0 \mathbf{1} + d_1 B + \cdots + d_{n-1} B^{\circ(n-1)}) - B^{\circ(n+M)},$$

for any square matrix B of arbitrary order, where t is a real scalar and n is the length of the tuple $\mathbf{d} = (d_0, \dots, d_{n-1})$. It suffices to show that $p_t[A; M-N, \mathbf{c}] \geq 0$ for all $t \geq \mathcal{C}(\mathbf{c}; z^M; N, \rho)$ and all $A = (a_{ij}) \in \mathcal{P}_N(\overline{D}(0,\rho))$.

By a lemma of FitzGerald and Horn [13, Lemma 2.1], if $\mathbf{u} \in \mathbb{C}^N$ is defined to equal $(a_{iN}/\sqrt{a_{NN}})_{i=1}^N$ when $a_{NN} \neq 0$, and is the zero vector otherwise, then $A - \mathbf{u}\mathbf{u}^*$ is positive semidefinite and its final column and row are both zero. Furthermore, the principal minor

$$\begin{vmatrix} a_{ii} & a_{iN} \\ \overline{a_{iN}} & a_{NN} \end{vmatrix} = a_{ii} a_{NN} - |a_{iN}|^2 \geq 0,$$

so the entries of $\mathbf{u}\mathbf{u}^*$ lie in $\overline{D}(0,\rho)$. Since

$$f(z) - f(w) = \int_0^1 (z-w) f'(\lambda z + (1-\lambda) w) \, \mathrm{d}\lambda \qquad (z, w \in \mathbb{C})$$

for any entire function f, it follows that

$$p_t[A; M-N, \mathbf{c}] = p_t[\mathbf{u}\mathbf{u}^*; M-N, \mathbf{c}] + \int_0^1 (A - \mathbf{u}\mathbf{u}^*) \circ M p_{t/M}[\lambda A + (1-\lambda)\mathbf{u}\mathbf{u}^*; M-N, \mathbf{c}'] \, \mathrm{d}\lambda, \tag{3.16}$$

where the $(N-1)$-tuple $\mathbf{c}' := (c_1, 2c_2, \dots, (N-1)c_{N-1})$. Now, since $A - \mathbf{u}\mathbf{u}^*$ has last row and column both zero, the integrand in (3.16) is positive semidefinite if the matrix $p_{t/M}[A_\lambda; M-N, \mathbf{c}']$ is, where $A_\lambda \in \mathcal{P}_{N-1}(\overline{D}(0,\rho))$ is obtained by deleting the final row and column of $\lambda A + (1-\lambda)\mathbf{u}\mathbf{u}^*$. Thus, if we show the inequality

$$\mathcal{C}(\mathbf{c}; z^M; N, \rho) \geq M \cdot \mathcal{C}(\mathbf{c}'; z^{M-1}; N-1, \rho), \tag{3.17}$$

then both terms in Equation (3.16) are positive semidefinite for all $t \geq \mathcal{C}(\mathbf{c}; z^M; N, \rho)$, by the induction hypothesis, and this gives the result.

Finally, we prove (3.17): note that

$$\begin{aligned} M \cdot \mathcal{C}(\mathbf{c}'; z^{M-1}; N-1, \rho) &= M \sum_{j=0}^{N-2} \binom{M-1}{j}^2 \binom{M-1-j-1}{N-1-j-1}^2 \frac{\rho^{M-1-j}}{(j+1)c_{j+1}} \\ &= \sum_{j=0}^{N-2} \binom{M}{j+1}^2 \binom{M-(j+1)-1}{N-(j+1)-1}^2 \frac{\rho^{M-(j+1)}}{c_{j+1}} \cdot \frac{j+1}{M} \\ &= \sum_{j=1}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j} \cdot \frac{j}{M} \\ &\leq \mathcal{C}(\mathbf{c}; z^M; N, \rho). \end{aligned}$$

Thus (2) implies (1), and so concludes the proof of the theorem. □

Remark 3.6. Theorem 1.1 shows further that, for any subset $K \subset \mathbb{C}$ satisfying

$$(0, \rho) \subset K \subset \overline{D}(0, \rho),$$

the polynomial $f(z) := c_0 + c_1 z + \cdots + c_{N-1} z^{N-1} + c_M z^M$ preserves positivity on $\mathcal{P}_N(K)$ if and only if $(c_0, \ldots, c_{N-1}, c_M)$ satisfies Theorem 1.1(2). More generally, given any class $\mathcal{P}$ of matrices such that $\mathcal{P}_N^1((0,\rho)) \subset \mathcal{P} \subset \mathcal{P}_N(\overline{D}(0,\rho))$, Theorem 1.1 implies that $f[-]$ preserves positivity on $\mathcal{P}$ if and only if $(c_0, \ldots, c_{N-1}, c_M)$ satisfies Theorem 1.1(2). Similarly, Theorem 1.1 shows the surprising result that to preserve positivity on all of $\mathcal{P}_N(\overline{D}(0,\rho))$ is equivalent to preserving positivity on the much smaller subset of real rank-one matrices $\mathcal{P}_N^1((0,\rho))$.

3.3. **Positivity preservers: sufficient conditions.** At this point, extensions of Theorem 1.1 to more general classes of functions are within reach. We first introduce some notation.

Definition 3.7. Given $K \subset \mathbb{C}$, functions g, $h : K \to \mathbb{C}$, and a set of positive semidefinite matrices $\mathcal{P} \subset \bigcup_{N=1}^\infty \mathcal{P}_N(K)$, let $\mathcal{C}(h; g; \mathcal{P})$ be the smallest real number such that

$$g[A] \leq \mathcal{C}(h; g; \mathcal{P}) \cdot h[A], \qquad \forall A \in \mathcal{P}. \tag{3.18}$$

In other words, the constant $\mathcal{C}(h; g; \mathcal{P})$ is the *extreme critical value* of the family of linear pencils $\{g[A] - \mathbb{R}h[A] : A \in \mathcal{P}\}$. If $h_{\mathbf{c}}(z) := c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$ is a polynomial with coefficients $\mathbf{c} := (c_0, \ldots, c_{n-1})$, then we let

$$\mathcal{C}(\mathbf{c}; g; \mathcal{P}) := \mathcal{C}(h_{\mathbf{c}}; g; \mathcal{P})$$

to simplify the notation. Similarly, if $\mathcal{P} = \mathcal{P}_N(\overline{D}(0,\rho))$ for some $\rho \in (0,\infty)$ and an integer $N \geq 1$, then we let

$$\mathcal{C}(h; g; N, \rho) := \mathcal{C}(h; g; \mathcal{P}_N(\overline{D}(0,\rho))).$$

Finally, let

$$\mathcal{C}(\mathbf{c}; g; N, \rho) := \mathcal{C}(c_0 + \cdots + c_{n-1} z^{n-1}; g; \mathcal{P}_N(\overline{D}(0,\rho)))$$

for $\rho \in (0,\infty)$ and integers n, $N \geq 1$.

Note that the notation introduced in Definition 3.7 is consistent with the notation used in the previous sections of the paper. Also, note for future use that the constant $\mathcal{C}(h; g; \mathcal{P})$ is non-increasing in the last argument, as well as $\mathbb{R}_+$-subadditive in the second argument:

$$\mathcal{C}(h; \lambda_1 g_1 + \cdots + \lambda_n g_n; \mathcal{P}) \leq \sum_{j=1}^{n} \lambda_j \mathcal{C}(h; g_j; \mathcal{P}), \qquad \forall \lambda_1, \ldots, \lambda_n \geq 0. \tag{3.19}$$

We now show how Theorem 1.1 can naturally be extended to arbitrary polynomials.

Corollary 3.8. *Fix a bounded subset $K \subset \mathbb{C}$, and integers $N \geq 1$ and $M \geq 0$. There exists a universal constant $\mathfrak{h}_{N,M}(K) > 0$ depending only on N, M, and K, with the following property: for any integer $R \geq 0$ and any polynomial $f(x) = x^R \sum_{k=0}^{N+M} c_k x^k$ with real coefficients such that*

1. *$c_0, \ldots, c_{N-1} > 0$, and*
2. *$\min\{c_k : 0 \leq k \leq N-1\} \geq \mathfrak{h}_{N,M}(K) \cdot \max\{|c_l| : c_l < 0\}$,*

we have $f[-] : \mathcal{P}_N(K) \to \mathcal{P}_N(\mathbb{C})$.

Proof. Without loss of generality, we may assume $R = 0$; the general case follows from the Schur product theorem. Now fix $\rho > 0$ such that $K \subset \overline{D}(0, \rho)$, and let

$$\mathfrak{h}_{N,M}(K) := \sum_{m=0}^{M} \mathcal{C}((1, \ldots, 1); z^{N+m}; N, \rho) \in (0, \infty), \tag{3.20}$$

where the first argument has N ones. Now given $A \in \mathcal{P}_N(K)$, one finds

$$f[A] = \sum_{j=0}^{N-1} c_j A^{\circ j} + \sum_{j=N}^{N+M} c_j A^{\circ j} \geq \min_{0 \leq k \leq N-1} c_k \sum_{j=0}^{N-1} A^{\circ j} + \min_{l \geq N : c_l < 0} c_l \sum_{j=N}^{N+M} A^{\circ j}.$$

The result now follows immediately from Equation (3.19). $\square$

Next we show how our main result extends from polynomial to analytic functions.

Proof of Theorem 1.3. Without loss of generality, we may assume that $c_M \geq 0$ for all $M \geq N$, and, via a standard approximation argument, that the series in the statement is analytic in the disc $D(0, \rho + \epsilon)$, where $\epsilon > 0$.

The first part is immediate from Theorem 1.1 and Equation (3.19). To establish the bound (1.3), note first that, for $0 \leq j \leq N-1$, we have

$$\begin{aligned}\binom{M}{j}\binom{M-j-1}{N-j-1} &= \frac{M!}{j!(M-N)!(N-j-1)!(M-j)} \\ &\leq \frac{M!}{j!(M-N+1)!(N-j-1)!} = \binom{N-1}{j}\binom{M}{N-1}.\end{aligned}$$

Using the above analysis and Tonelli's theorem, we compute:

$$\begin{aligned}
&\sum_{M=N}^{\infty} c_M \mathcal{C}(\mathbf{c}; z^M; N, \rho) \\
&= \sum_{j=0}^{N-1} \frac{\rho^{N-j}}{c_j} \sum_{M=N}^{\infty} c_M \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \rho^{M-N} \\
&\leq \frac{1}{(N-1)!^2} \sum_{j=0}^{N-1} \binom{N-1}{j}^2 \frac{\rho^{N-j}}{c_j} \sum_{M=N}^{\infty} c_M \rho^{M-N} \prod_{k=2}^{N} (M-N+k)^2 \\
&\leq \frac{1}{(N-1)!^2} \sum_{j=0}^{N-1} \binom{N-1}{j}^2 \frac{\rho^{N-j}}{c_j} \sum_{M=N}^{\infty} c_M \rho^{M-N} \prod_{k=2}^{N} \frac{(2(M-N)+k+1)(2M-N+k)}{2} \\
&= \frac{2^{-(N-1)}}{(N-1)!^2} \sum_{j=0}^{N-1} \binom{N-1}{j}^2 \frac{\rho^{N-j-1}}{c_j} \sum_{M=N}^{\infty} c_M \prod_{k=0}^{2N-3} (2M-k) \cdot (\sqrt{\rho})^{2M-2N+2}.
\end{aligned}$$

Notice that g_2 is analytic on $D(0, \sqrt{\rho+\epsilon})$, since g is analytic on $D(0, \rho+\epsilon)$. Therefore the inner sum is precisely the $(2N-2)$th derivative of $g_2(z) = g_+(z^2)$, evaluated at $z = \sqrt{\rho}$. This concludes the proof. $\square$

Remark 3.9. The quantity $g_2^{(2N-2)}(\sqrt{\rho})$ can be written in terms of the derivatives of g_+ at $z = \rho$; it may be shown by induction that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(g_+(x^2)\Big) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!} (2x)^{n-2k} g_+^{(n-k)}(x^2) \qquad (n \geq 0). \tag{3.21}$$

This shows that one has explicit bounds for $\mathcal{C}(\mathbf{c}; g; N, \rho) \leq \mathcal{C}(\mathbf{c}; g_+; N, \rho)$ in terms of the derivatives of g_+ at ρ.

We conclude this part by showing how Theorem 1.1 yields an asymptotically sharp bound for the matrix cube problem for Hadamard powers.

Proof of Corollary 1.5. Note that $\mathcal{U}[\eta] \subset \mathcal{P}_N(\mathbb{C})$ if and only if

$$A_0 - \eta \sum_{m=1}^{M+1} A_m = c_0 \mathbf{1}_{N \times N} + A + \cdots + c_{N-1} A^{\circ(N-1)} - \eta(A^{\circ N} + \cdots + A^{\circ(N+M)}) \in \mathcal{P}_N(\mathbb{C}).$$

Thus the first implication follows by Theorem 1.3. The other implication follows by setting $u_1 = \cdots = u_M = 0$ in Equation (1.5) and applying Theorem 1.1.

It remains to show the asymptotics in (1.8). For this it suffices to show that

$$\lim_{N \to \infty} \frac{\mathcal{C}(\mathbf{c}; z^{N+m}; N, \rho)}{\mathcal{C}(\mathbf{c}; z^{N+M}; N, \rho)} = 0 \qquad (0 \leq m \leq M-1). \tag{3.22}$$

In turn, to show (3.22) we first fix N and ρ, and write out each summand in the numerator and denominator of (3.22) as follows:

$$a(m,j) := \binom{N+m}{j}^2 \binom{N+m-j-1}{N-j-1}^2 \frac{\rho^{N+m-j}}{c_j}.$$

Next we bound the ratios of the summands for fixed N, m, M, and ρ, and for $0 \leq j \leq N-1$:

$$\begin{aligned}\frac{a(m,j)}{a(M,j)} &= \Big(\frac{(N+m)!M!(N+M-j)}{m!(N+M)!(N+m-j)}\Big)^2 \rho^{m-M} \\ &= \frac{1}{(N+M)^2 \cdots (N+m+1)^2} \Big(\frac{M!}{m!} \cdot \frac{N+M-j}{N+m-j}\Big)^2 \rho^{m-M} \\ &\leq \frac{1}{N^{2(M-m)}} \Big(\frac{M!}{m!} \cdot \frac{N+M-j}{N+m-j}\Big)^2 \rho^{m-M} \\ &\leq \frac{1}{N^{2(M-m)}} \Big(\frac{(M+1)!}{(m+1)!}\Big)^2 \rho^{m-M},\end{aligned}$$

where the final step follows from the observation that if $M > m \geq 0$ are fixed, and $N - j = \alpha \geq 1$, then $\frac{\alpha+M}{\alpha+m} = 1 + \frac{M-m}{\alpha+m}$ has its global maximum at $\alpha = 1$. Now let

$$b(m, M, \rho) := \Big(\frac{(M+1)!}{(m+1)!}\Big)^2 \rho^{m-M},$$

so that $a(m,j) \leq a(M,j) b(m,M,\rho) N^{-2(M-m)}$ for $0 \leq j \leq N-1$. Hence

$$0 \leq \frac{\mathcal{C}(\mathbf{c}; z^{N+m}; N, \rho)}{\mathcal{C}(\mathbf{c}; z^{N+M}; N, \rho)} \leq b(m, M, \rho) N^{-2(M-m)} \to 0$$

as $N \to \infty$, as required. □

3.4. **Case study: 2×2 matrices.** Remark that Theorem 1.1 holds for any integer $N \geq 1$. When $N = 2$, it is possible to prove a characterization result for polynomials preserving positivity on $\mathcal{P}_2$, for a more general family of polynomials than in Theorem 1.1. Along these lines, we conclude the present section with the following result.

Theorem 3.10. *Given non-negative integers $m < n < p$, let $f(x) = c_m x^m + c_n x^n + c_p x^p$, with c_m and c_n both non-zero. The following are equivalent.*

(1) *The entrywise function $f[-]$ preserves positivity on $\mathcal{P}_2([0,1])$.*
(2) *The entrywise function $f[-]$ preserves positivity on $\mathcal{P}_2^1([0,1])$.*
(3) *c_m, $c_n > 0$, and*

$$c_p \geq \frac{-c_m c_n (n-m)^2}{c_m (p-m)^2 + c_n (p-n)^2}.$$

Note that if $c_p < 0$ then such an entrywise function does not preserve positivity on $\mathcal{P}_3([0,1])$.

In the special case $m = 0$ and $n = 1$, note that the bound $\frac{-c_m c_n (n-m)^2}{c_m (p-m)^2 + c_n (p-n)^2}$ reduces to the constant $-\mathcal{C}(\mathbf{c}; p; 2, 1)^{-1}$.

Proof. Clearly (1) implies (2). To see why (2) implies (3), note that c_m, $c_n > 0$ by Lemma 2.4. Now let u_1, $u_2 \in [0,1]$ be distinct and let $f_m(u_1, u_2) := (u_1^m - u_2^m)/(u_1 - u_2)$

denote the divided difference of their mth powers. With $\mathbf{u} = (u_1, u_2)^T$,

$$\begin{aligned}0 \leq \det f[\mathbf{u}\mathbf{u}^T] &= f(u_1^2)f(u_2^2) - f(u_1u_2)^2 \\ &= c_m c_n (u_1u_2)^{2m}(u_2^{n-m} - u_1^{n-m})^2 + c_m c_p (u_1u_2)^{2m}(u_2^{p-m} - u_1^{p-m})^2 \\ &\quad + c_n c_p (u_1u_2)^{2n}(u_2^{p-n} - u_1^{p-n})^2,\end{aligned}$$

which implies that

$$c_p \geq \frac{-c_m c_n f_{n-m}(u_1, u_2)^2}{c_m f_{p-m}(u_1, u_2)^2 + c_n (u_1u_2)^{2(n-m)} f_{p-n}(u_1, u_2)^2}.$$

Letting $u_1 \to 1 = u_2$ yields (3).

Finally, suppose (3) holds. Then (1) holds if and only if $f(xy)^2 \leq f(x^2)f(y^2)$ for all x, $y \in [0,1]$ and f is non-decreasing and non-negative on $[0,1]$; see, for example, [18, Theorem 2.5]. By [38, Exercise 2.4.4], the first of these conditions is satisfied if and only if the function

$$\Psi_f(x) := x(f''(x)f(x) - f'(x)^2) + f(x)f'(x) \geq 0$$

for all $x \in (0,1)$. A short calculation gives that

$$\Psi_f(x) = x^{m+n-1}\big(c_m c_n (n-m)^2 + c_m c_p (p-m)^2 x^{p-n} + c_n c_p (p-n)^2 x^{p-m}\big),$$

and so

$$\Psi_f(x) \geq 0 \iff c_p \geq \frac{-c_m c_n (n-m)^2}{c_m (p-m)^2 x^{p-m} + c_n (p-n)^2 x^{p-n}}.$$

The final term has its infimum on $(0,1)$ when $x = 1$, so we obtain the first condition. Next, note that f is non-decreasing on $[0,1]$ if and only if

$$\begin{aligned}& f'(x) = m c_m x^{m-1} + n c_n x^{n-1} + p c_p x^{p-1} \geq 0 \quad \forall x \in (0,1) \\ \iff & c_p \geq \frac{-m c_m x^{m-p} - n c_n x^{n-p}}{p} \quad \forall x \in (0,1) \\ \iff & c_p \geq \frac{-m c_m - n c_n}{p}.\end{aligned}$$

Thus, it suffices to show that

$$\frac{m c_m + n c_n}{p} \geq \frac{c_m c_n (n-m)^2}{c_m (p-m)^2 + c_n (p-n)^2},$$

but this holds because

$$c_m c_n (m(p-n)^2 + n(p-m)^2 - p(n-m)^2) = c_m c_n (m+n)(p-m)(p-n) \geq 0.$$

Thus (3) implies that f is non-decreasing on $[0,1]$. In turn, this implies that f is non-negative on $[0,1]$, since $f(x) \geq f(0) \geq 0$ for $x \in [0,1]$. Hence (3) implies (1).

The final assertion is an immediate consequence of Lemma 2.4. $\square$

4. Rayleigh quotients

Recall from Theorem 1.6 that Theorem 1.1 can be reformulated as an extremal problem, involving the boundedness of the generalized Rayleigh quotient for $\sum_{j=0}^{N-1} c_j A^{\circ j}$ and $A^{\circ M}$, taken over all matrices $A \in \mathcal{P}_N(\overline{D}(0,\rho))$. We now consider an alternate approach to proving Theorem 1.1, by first considering the analogue of Theorem 1.6 for a single matrix. In this case a sharp bound may be obtained as follows.

Proposition 4.1. *Fix integers $N \geq 1$ and $M \geq 0$, positive scalars $c_0, \ldots, c_{N-1} > 0$, and a non-zero matrix $A \in \mathcal{P}_N(\mathbb{C})$. Then, with the notation as in Theorem 1.6 and Definition 3.7, $\mathcal{K}(A) \subset \ker A^{\circ M}$, and the corresponding extreme critical value $\mathcal{C}(\mathbf{c}; z^M; A)$ is finite:*

$$\mathcal{C}(\mathbf{c}; z^M; A)^{-1} = \min_{\mathbf{v} \in S^{2N-1} \cap \mathcal{K}(A)^{\perp}} \frac{\mathbf{v}^* \left(\sum_{j=0}^{N-1} c_j A^{\circ j} \right) \mathbf{v}}{\mathbf{v}^* A^{\circ M} \mathbf{v}},$$

where S^{2N-1} is the unit sphere in $\mathbb{C}^N$.

In particular,

$$\mathbf{v}^* \left(\sum_{j=0}^{N-1} c_j A^{\circ j} \right) \mathbf{v} \geq \mathcal{C}(\mathbf{c}; z^M; A)^{-1} \cdot \mathbf{v}^* A^{\circ M} \mathbf{v}$$

for all $\mathbf{v} \in \mathbb{C}^N$.

Note, moreover, that the minimum in Proposition 4.1 is attained for every non-zero matrix $A \in \mathcal{P}_N(\mathbb{C})$ and integer $M \geq 0$, and over a compact set that is independent of M.

The proof of Proposition 4.1, as well as a closed-form expression for the constant $\mathcal{C}(\mathbf{c}; z^M; A)$, is immediate from the following two results.

Proposition 4.2. *Given $N \geq 1$ and C, $D \in \mathcal{P}_N(\mathbb{C})$, the following are equivalent.*

1. *If $\mathbf{v}^* C \mathbf{v} = 0$ for some $\mathbf{v} \in \mathbb{C}^N$, then $\mathbf{v}^* D \mathbf{v} = 0$.*
2. $\ker C \subset \ker D$.
3. *There exists a smallest positive constant $\mathfrak{h}_{C,D}$ such that $\mathbf{v}^* C \mathbf{v} \geq \mathfrak{h}_{C,D} \cdot \mathbf{v}^* D \mathbf{v}$ for all $\mathbf{v} \in \mathbb{C}^N$.*

Moreover, if (1)–(3) hold and $D \neq 0$, then the constant is computable as an extremal generalized Rayleigh quotient, as follows:

$$\mathfrak{h}_{C,D}^{-1} = \sup_{\mathbf{v} \notin \ker D} \frac{\mathbf{v}^* D \mathbf{v}}{\mathbf{v}^* C \mathbf{v}} = \varrho(C^{\dagger/2} D C^{\dagger/2}) = \varrho(X^* C^{\dagger} X), \tag{4.1}$$

where $C^{\dagger/2}$ and $C^{\dagger} \in \mathcal{P}_N(\mathbb{C})$ denote respectively the square root of the Moore–Penrose inverse and the Moore–Penrose inverse of $C \in \mathcal{P}_N(\mathbb{C})$, $\varrho(-)$ denotes the spectral radius, and X is any matrix such that $D = XX^*$.

Proof. Clearly (3) implies (1). That (1) is equivalent to (2) is also immediate, given the following reasoning:

$$\begin{aligned} \mathbf{v}^* C \mathbf{v} = 0 \quad &\iff \quad \mathbf{v}^* C^{1/2} \cdot C^{1/2} \mathbf{v} = 0 \\ &\iff \quad C^{1/2} \mathbf{v} = 0 \quad \implies \quad C \mathbf{v} = 0 \implies \quad \mathbf{v}^* C \mathbf{v} = 0. \end{aligned}$$

We now show that (1) implies (3). Given a matrix C, denote its kernel and the orthogonal complement of its kernel by K_C and $K_C^\perp$, respectively. If $\mathbf{v} \in K_D$ then any choice of constant $\mathfrak{h}_{C,D}$ would suffice in (3), so we may restrict ourselves to obtaining such a constant when $\mathbf{v} \notin K_D$. Write $\mathbf{v} = \mathbf{v}_C + \mathbf{v}_C^\perp$, with $\mathbf{v}_C \in K_C$ and $\mathbf{v}_C^\perp \in K_C^\perp$, and note that $\mathbf{v}_C^\perp \neq 0$. Now compute:

$$\frac{\mathbf{v}^* C \mathbf{v}}{\mathbf{v}^* D \mathbf{v}} = \frac{(\mathbf{v}_C^\perp)^* C \mathbf{v}_C^\perp}{(\mathbf{v}_C^\perp)^* D \mathbf{v}_C^\perp} = \frac{(\mathbf{v}_C^\perp / \|\mathbf{v}_C^\perp\|)^* C (\mathbf{v}_C^\perp / \|\mathbf{v}_C^\perp\|)}{(\mathbf{v}_C^\perp / \|\mathbf{v}_C^\perp\|)^* D (\mathbf{v}_C^\perp / \|\mathbf{v}_C^\perp\|)}.$$

It follows by setting $\mathbf{w} := \mathbf{v}_C^\perp / \|\mathbf{v}_C^\perp\|$ that

$$\mathfrak{h}_{C,D} := \inf_{\mathbf{v} \notin K_D} \frac{\mathbf{v}^* C \mathbf{v}}{\mathbf{v}^* D \mathbf{v}} = \min_{\mathbf{w} \in S^{2N-1} \cap K_C^\perp} \frac{\mathbf{w}^* C \mathbf{w}}{\mathbf{w}^* D \mathbf{w}}, \tag{4.2}$$

where S^{2N-1} denotes the unit sphere in $\mathbb{C}^N$. The right-hand side is the minimizer of a continuous and positive function over a compact set, hence equals a positive real number. Thus $\mathfrak{h}_{C,D} > 0$, as desired.

We now establish Equation (4.1), by following the theory of the Kronecker normal form for a matrix pencil, as developed in [14, Chapter X, §6]. Because both sides of the last equality in Equation (4.1) remain unchanged under a unitary change of basis, we may assume that C is diagonal, say $C = \operatorname{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$ with $\lambda_1 \geq \cdots \geq \lambda_r > 0$. Pre-multiplying by $C^{1/2} = \operatorname{diag}(\lambda_1^{1/2}, \ldots, \lambda_r^{1/2}, 0, \ldots, 0)$ preserves $K_C^\perp \setminus \{0\}$, and therefore, setting $\mathbf{w} = C^{1/2} \mathbf{v}_C^\perp$,

$$\begin{aligned}
\mathfrak{h}_{C,D}^{-1} &= \sup_{\mathbf{v} \notin \ker D} \frac{\mathbf{v}^* D \mathbf{v}}{\mathbf{v}^* C \mathbf{v}} = \sup_{\mathbf{v}_C^\perp \in K_C^\perp \setminus \{0\}} \frac{(\mathbf{v}_C^\perp)^* D \mathbf{v}_C^\perp}{(\mathbf{v}_C^\perp)^* C \mathbf{v}_C^\perp} = \sup_{\mathbf{w} \in K_C^\perp \setminus \{0\}} \frac{\mathbf{w}^* C^{\dagger/2} D C^{\dagger/2} \mathbf{w}}{\mathbf{w}^* \mathbf{w}} \\
&= \sup_{\mathbf{w} \in S^{2N-1} \cap K_C^\perp} \mathbf{w}^* C^{\dagger/2} D C^{\dagger/2} \mathbf{w} \\
&\leq \sup_{\mathbf{w} \in S^{2N-1}} \mathbf{w}^* C^{\dagger/2} D C^{\dagger/2} \mathbf{w} = \varrho(C^{\dagger/2} D C^{\dagger/2}).
\end{aligned}$$

Moreover, any non-zero eigenvector of $C^{\dagger/2} D C^{\dagger/2}$ with non-zero eigenvalue has its last $n - r$ coordinates all equal to zero, since $C^{\dagger/2} = \operatorname{diag}(\lambda_1^{-1/2}, \ldots, \lambda_r^{-1/2}, 0, \ldots, 0)$. It follows that the unit-length eigenvectors corresponding to the eigenvalue $\varrho(C^{\dagger/2} D C^{\dagger/2})$ lie in $K_C^\perp$, thereby proving the result. The last equality in Equation (4.1) is an immediate consequence of the tracial property of the spectral radius. □

Lemma 4.3. *Fix integers $m \geq N \geq 1$ and $M \geq 0$, and matrices A, $B \in \mathcal{P}_N(\mathbb{C})$. The matrices $h(A, B) := B \circ \sum_{j=0}^{m-1} A^{\circ j}$ and $g(A, B) := B \circ A^{\circ M}$ are such that* $\ker h(A, B) \subset \ker g(A, B)$.

Proof. Suppose $\mathbf{v} \in \ker h(A, B)$, so that $\mathbf{v}^* h(A, B) \mathbf{v} = 0$. Since $B \circ A^{\circ j} \in \mathcal{P}_N(\mathbb{C})$, it follows that $\mathbf{v}^*(B \circ A^{\circ j})\mathbf{v} = 0$ and $(B \circ A^{\circ j})\mathbf{v} = 0$ for $0 \leq j \leq m - 1$; in particular, the lemma is true if $M < m$. If, instead, $M \geq N$, then apply Lemma 3.5, together with the fact that $D(C \circ B) = (DC) \circ B$ if B, C and D are square matrices, with D

diagonal, to see that

$$h(A,B)\mathbf{v} = 0 \quad \Longrightarrow \quad (A^{\circ j} \circ B)\mathbf{v} = 0 \quad (0 \leq j \leq N-1)$$
$$\Longrightarrow \quad \sum_{j=0}^{N-1} D_{M,j}(A)(A^{\circ j} \circ B)\mathbf{v} = 0 \quad \Longrightarrow \quad (A^{\circ M} \circ B)\mathbf{v} = 0. \ \square$$

Remark 4.4. Note that Theorem 1.6 does not hold in much greater generality, i.e., for more general functions g, h than the polynomials used in Lemma 4.3. For instance, fix $\delta > 0$ and consider continuous functions $g, h : [0,\delta) \to [0,\infty)$ such that $g(0) = h(0) = 0$ and $g(x), h(x) > 0$ on $(0,\delta)$. Given $N \geq 2$ and $t \in \mathbb{R}$, let the $N \times N$ matrix $A(t) := \operatorname{diag}(t, 1, \dots, 1)$. There need not exist a universal constant $\mathfrak{h} > 0$ such that

$$\mathbf{v}^T h[A(t)]\mathbf{v} \geq \mathfrak{h} \cdot \mathbf{v}^T g[A(t)]\mathbf{v} \qquad \forall t \in (0,\delta), \ \mathbf{v} \in \mathbb{R}^N. \tag{4.3}$$

To see this, note that $A(t)$, $h[A(t)]$ and $g[A(t)]$ are positive definite for $t \in (0,\delta)$ and singular at $t = 0$. Thus, by standard facts about the generalized Rayleigh quotient, if (4.3) holds, then

$$\begin{aligned}\mathfrak{h} \leq \inf_{t\in(0,\delta)} \inf_{\mathbf{v}\in\mathbb{R}^N\setminus\{0\}} \frac{\mathbf{v}^T h[A(t)]\mathbf{v}}{\mathbf{v}^T g[A(t)]\mathbf{v}} &= \inf_{t\in(0,\delta)} \lambda_{\min}(g[A(t)]^{-1/2} h[A(t)] g[A(t)]^{-1/2}) \\ &= \inf_{t\in(0,\delta)} \min\{1, h(t)/g(t)\},\end{aligned}$$

where $\lambda_{\min}(C)$ denotes the smallest eigenvalue of the positive semidefinite matrix C. If $h(t) = t^a g(t)$ for some $a > 0$, then this infimum, and so $\mathfrak{h}$, equals zero. Note, however, that such a case does not occur in Theorem 1.6.

We now observe that Proposition 4.1 and the explicit formula for the extreme critical value of the matrix pencil, as given in Equation (4.1), allow us to provide a closed-form expression for the constant $\mathcal{C}(\mathbf{c}; z^M; A)$ for rank-one matrices.

Corollary 4.5. *Given integers $N \geq 1$ and $M \geq 0$, positive scalars $c_0, \dots, c_{N-1} > 0$, and a matrix $A = \mathbf{u}\mathbf{u}^* \in \mathcal{P}_N^1(\mathbb{C}) \setminus \{\mathbf{0}_{N\times N}\}$, the identity*

$$\mathcal{C}(\mathbf{c}; z^M; A) = (\mathbf{u}^{\circ M})^* \left(\sum_{j=0}^{N-1} c_j \mathbf{u}^{\circ j} (\mathbf{u}^{\circ j})^* \right)^{\dagger} \mathbf{u}^{\circ M}$$

holds. In particular, given $\rho > 0$,

$$\mathcal{C}(\mathbf{c}; z^M; \rho \mathbf{1}_{N\times N}) = \rho^M \left(\sum_{j=0}^{N-1} c_j \rho^j \right)^{-1} \leq \mathcal{C}(\mathbf{c}; z^M; N, \rho),$$

with equality if and only if $N = 1$.

Proof. Applying Proposition 4.1 and Equation (4.1) with $C = h_{\mathbf{c}}[A] := \sum_{j=0}^{N-1} c_j A^{\circ j}$ and $D = A^{\circ M} = \mathbf{u}^{\circ M}(\mathbf{u}^{\circ M})^*$, we obtain

$$\mathcal{C}(\mathbf{c}; z^M; A) = \varrho(\mathbf{v}\mathbf{v}^*), \quad \text{where} \quad \mathbf{v} = h_{\mathbf{c}}[\mathbf{u}\mathbf{u}^*]^{\dagger/2} \mathbf{u}^{\circ M}.$$

Now, it is well known that the unique non-zero eigenvalue of $\mathbf{v}\mathbf{v}^*$ equals

$$\mathbf{v}^*\mathbf{v} = (\mathbf{u}^{\circ M})^* h_{\mathbf{c}}[A]^\dagger \mathbf{u}^{\circ M} = (\mathbf{u}^{\circ M})^* \left(\sum_{j=0}^{N-1} c_j \mathbf{u}^{\circ j} (\mathbf{u}^{\circ j})^* \right)^\dagger \mathbf{u}^{\circ M},$$

which shows the first assertion. For the next, set $\mathbf{u} = \sqrt{\rho}(1, \ldots, 1)^*$, and use the fact that $(\alpha \mathbf{u}\mathbf{u}^*)^\dagger = \alpha^{-1}(\mathbf{u}^*\mathbf{u})^{-2}(\mathbf{u}\mathbf{u}^*)$ for non-zero α and $\mathbf{u}$, to obtain that

$$\mathcal{C}(\mathbf{c}; z^M; \rho \mathbf{1}_{N\times N}) = \rho^M \left(\sum_{j=0}^{N-1} c_j \rho^j \right)^{-1}.$$

It remains to show the last inequality. When $N = 1$,

$$\mathcal{C}(\mathbf{c}; z^M; N, \rho) \sum_{j=0}^{N-1} c_j \rho^j = \frac{\rho^M}{c_0} \cdot c_0 = \rho^M.$$

If instead $N > 1$, then

$$\mathcal{C}(\mathbf{c}; z^M; N, \rho) \sum_{j=0}^{N-1} c_j \rho^j > \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \rho^M > \rho^M. \qquad \square$$

Remark 4.6. If $A = \mathbf{u}\mathbf{u}^*$, with $\mathbf{u}$ having pairwise-distinct entries, then $h_{\mathbf{c}}[A]$ is the sum of N rank-one matrices with linearly independent column spaces, and hence is non-singular. In particular, it is possible to write $\mathcal{C}(\mathbf{c}; z^M; A)$ for such A in an alternate fashion: if $h_{\mathbf{c}}(z) = \sum_{j=0}^{N-1} c_j z^j$, where $c_0, \ldots, c_{N-1} > 0$ then

$$\mathcal{C}(\mathbf{c}; z^M; A) = (\mathbf{u}^{\circ M})^* h_{\mathbf{c}}[\mathbf{u}\mathbf{u}^*]^{-1} \mathbf{u}^{\circ M} = 1 - \det h_{\mathbf{c}}[\mathbf{u}\mathbf{u}^*]^{-1} \det \begin{pmatrix} h_{\mathbf{c}}[\mathbf{u}\mathbf{u}^*] & \mathbf{u}^{\circ M} \\ (\mathbf{u}^{\circ M})^* & 1 \end{pmatrix}.$$

For the optimization-oriented reader, we now restate the main result as an extremal problem that follows immediately from Theorem 1.6 and Proposition 4.1:

$$\begin{aligned} &\inf_{A \in \mathcal{P}_N^1((0,\rho))} \min_{\mathbf{v} \in S^{2N-1} \cap \mathcal{K}(A)^\perp} \frac{\mathbf{v}^* \left(\sum_{j=0}^{N-1} c_j A^{\circ j} \right) \mathbf{v}}{\mathbf{v}^* A^{\circ M} \mathbf{v}} \\ &= \inf_{A \in \mathcal{P}_N(\overline{D}(0,\rho))} \min_{\mathbf{v} \in S^{2N-1} \cap \mathcal{K}(A)^\perp} \frac{\mathbf{v}^* \left(\sum_{j=0}^{N-1} c_j A^{\circ j} \right) \mathbf{v}}{\mathbf{v}^* A^{\circ M} \mathbf{v}} \\ &= \left(\sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j} \right)^{-1}, \end{aligned} \tag{4.4}$$

or, equivalently,

$$\sup_{A \in \mathcal{P}_N^1((0,\rho))} \mathcal{C}(\mathbf{c}; z^M; A) = \sup_{A \in \mathcal{P}_N(\overline{D}(0,\rho))} \mathcal{C}(\mathbf{c}; z^M; A) = \mathcal{C}(\mathbf{c}; z^M; N, \rho), \tag{4.5}$$

where $\mathcal{C}(\mathbf{c}; z^M; A) = \varrho(h_{\mathbf{c}}[A]^{\dagger/2} A^{\circ M} h_{\mathbf{c}}[A]^{\dagger/2})$, with $h_{\mathbf{c}}(z) = \sum_{j=0}^{N-1} c_j z^j$.

Remark 4.7. Note from the previous line that $A \mapsto \mathcal{C}(\mathbf{c}; z^M; A)$ is continuous on the subset of the cone $\mathcal{P}_N(\mathbb{C})$ where $\det h_{\mathbf{c}}[A] \neq 0$. The obstacle to establishing (4.4) and (4.5), and so proving Theorems 1.1 and 1.6, resides in the fact that this function is not continuous on the whole of $\mathcal{P}_N(\mathbb{C})$ for $N > 1$. In particular, it is not continuous at the matrix $A = \rho \mathbf{1}_{N \times N} \in \mathcal{P}_N^1(\overline{D}(0,\rho))$, as shown in the calculations for Equation (3.12) and Corollary 4.5 above. However, these calculations also reveal that the sharp constant $\mathcal{C}(\mathbf{c}; z^M; N, \rho)$ is obtained by taking the supremum of the Rayleigh quotient over the one-parameter family of rank-one matrices

$$\{\rho \mathbf{u}(t)\mathbf{u}(t)^T : \mathbf{u}(t) := (1-t, \dots, 1-Nt)^T, \ t \in (0, 1/N)\}.$$

5. The simultaneous kernels

Prompted by the variational approach of the previous section, a description of the kernel $\mathcal{K}(A) = \ker(c_0 \mathbf{1}_{N \times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)})$ for any $A \in \mathcal{P}_N(\mathbb{C})$ is in order. As we prove below, this kernel does not depend on the choice of scalars $c_j > 0$, and coincides with the simultaneous kernel

$$\bigcap_{n \geq 0} \ker A^{\circ n}$$

of the Hadamard powers of A. A refined structure of the matrix A, based on an analysis of the kernels of iterated Hadamard powers, is isolated in the first part of this section.

5.1. Stratifications of the cone of positive semidefinite matrices.

We introduce a novel family of stratifications of the cone $\mathcal{P}_N(\mathbb{C})$, that are induced by partitions of the set $\{1, \dots, N\}$. Each stratification is subjacent to a block decomposition of a positive semidefinite $N \times N$ matrix, with diagonal blocks having rank one. In addition, the blocks exhibit a remarkable homogeneity with respect to subgroups of the group $\mathbb{C}^\times$, the multiplicative group of non-zero complex numbers.

Theorem 5.1. *Fix a subgroup $G \subset \mathbb{C}^\times$, an integer $N \geq 1$, and a non-zero matrix $A \in \mathcal{P}_N(\mathbb{C})$.*

(1) *Suppose $\{I_1, \dots, I_k\}$ is a partition of $\{1, \dots, N\}$ satisfying the following two conditions.*
   (a) *Each diagonal block A_{I_j} of A is a submatrix having rank at most one, and $A_{I_j} = \mathbf{u}_j \mathbf{u}_j^*$ for a unique $\mathbf{u}_j \in \mathbb{C}^{|I_j|}$ with first entry $\mathbf{u}_{j,1} \in [0, \infty)$.*
   (b) *The entries of each diagonal block A_{I_j} lie in a single G-orbit.*

   Then there exists a unique matrix $C = (c_{ij})_{i,j=1}^k$ such that $c_{ij} = 0$ unless $\mathbf{u}_i \neq 0$ and $\mathbf{u}_j \neq 0$, and A is a block matrix with

$$A_{I_i \times I_j} = c_{ij} \mathbf{u}_i \mathbf{u}_j^* \qquad (1 \leq i, j \leq k).$$

   Moreover, the entries of each off-diagonal block of A also lie in a single G-orbit. Furthermore, the matrix $C \in \mathcal{P}_k(\overline{D}(0,1))$, and the matrices A and C have equal rank.

(2) *If the following condition (c) is assumed as well as (a) and (b), then such a partition $\{I_1, \dots, I_k\}$ exists and is unique up to relabelling of the indices.*
   (c) *The diagonal blocks of A have maximal size, i.e., each diagonal block is not contained in a larger diagonal block that has rank one.*

(3) *Suppose (a)–(c) hold and $G = \mathbb{C}^\times$. Then the off-diagonal entries of C lie in the open disc $D(0,1)$.*

In particular, given any non-zero matrix $A \in \mathcal{P}_N(\mathbb{C})$, there exists a unique partition $\{I_1, \ldots, I_k\}$ of $\{1, \ldots, N\}$ having minimal size k, unique vectors $\mathbf{u}_j \in \mathbb{C}^{|I_j|}$ with $\mathbf{u}_{j,1} \in [0,\infty)$, and a unique matrix $C \in \mathcal{P}_k(\overline{D}(0,1))$, such that A is a block matrix with $A_{I_i \times I_j} = c_{ij} \mathbf{u}_i \mathbf{u}_j^*$ whenever $1 \leq i, j \leq k$, and A has rank at most k.

Proof.

(1) Suppose $1 \leq i \neq j \leq k$, and $1 \leq l < l' < m \leq N$, with l, $l' \in I_i$ and $m \in I_j$; the submatrix

$$B := A_{\{l,l',m\}} = \begin{pmatrix} a & ag & b \\ a\overline{g} & a|g|^2 & c \\ \overline{b} & \overline{c} & d \end{pmatrix},$$

where a, $d \geq 0$, $g \in G$, and b, $c \in \mathbb{C}$.

We claim that $c \in b \cdot G$, and that the minor $\begin{pmatrix} a & b \\ a\overline{g} & c \end{pmatrix}$ is singular. Now,

$$0 \leq \det B = -a(|c|^2 + |b|^2|g|^2 - 2\operatorname{Re}(\overline{b}cg)) = -a|c - b\overline{g}|^2,$$

so either $a = 0$, in which case $b = c = 0$, by the positivity of B, or $c = b\overline{g}$. This proves the claim.

Applying this result repeatedly shows that every 2×2 minor of the block matrix $A_{I_i \cup I_j}$, with at least two entries in the same block, is singular, and the entries of any off-diagonal block lie in a single G-orbit. Thus there exists a unique Hermitian matrix C such that all assertions in the first part hold, except for possibly the claim that $C \in \mathcal{P}_k(\overline{D}(0,1))$.

Fix any vector $\mathbf{v} \in \mathbb{C}^k$, and choose vectors $\mathbf{w}_j \in \mathbb{C}^{|I_j|}$ such that $\mathbf{u}_j^* \mathbf{w}_j = v_j$ if $\mathbf{u}_j \neq 0$, and arbitrarily otherwise. Define $\mathbf{w} := (\mathbf{w}_1^*, \ldots, \mathbf{w}_k^*)^* \in \mathbb{C}^N$, and note that

$$0 \leq \mathbf{w}^* A \mathbf{w} = \sum_{i,j=1}^k \mathbf{w}_i^* c_{ij} \mathbf{u}_i \mathbf{u}_j^* \mathbf{w}_j = \sum_{i,j=1}^k \overline{v_i} c_{ij} v_j = \mathbf{v}^* C \mathbf{v}, \tag{5.1}$$

so $C \in \mathcal{P}_k(\mathbb{C})$. Consequently, the entries of C are in $\overline{D}(0,1)$, by a positivity argument, because the diagonal entries of C are all 0 or 1.

It remains to show that A and C have the same rank. Let $J := \{j \in \{1, \ldots, k\} : A_{I_j} \neq 0\}$ be the set of indices of non-zero diagonal blocks of A, let $I := \cup_{j \in J} I_j$, and, for all $j \in J$, let j' be the least element of I_j. Define a linear map $\pi : \mathbb{C}^J \to \mathbb{C}^I$ by letting $\pi(\varepsilon_j) := \mathbf{u}_{j',1}^{-1} \varepsilon'_{j'}$ for all $j \in J$, where ε_j and $\varepsilon'_{j'}$ are the standard basis elements in $\mathbb{C}^J$ and $\mathbb{C}^I$ labelled by j and j', respectively. Then

$$\mathbf{v}^* C_J \mathbf{v} = \pi(\mathbf{v})^* A_I \pi(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbb{C}^J,$$

so π restricts to a linear isomorphism between $\ker C_J$ and $\ker A_I \cap \pi(\mathbb{C}^J)$. Since the matrices A and A_I have equal rank, as have C and C_J, it follows that the ranks of A and C are equal; note that $\ker A_I \cap \pi(\mathbb{C}^J)$ is naturally isomorphic to $\ker C_J$, and $\{\mathbf{u}'_j : j \in J\}^\perp$ is a subset of $\ker A_I$ which has trivial intersection with $\pi(\mathbb{C}^J)$, where $\mathbf{u}'_j$ is the natural embedding of $\mathbf{u}_j$ in $\mathbb{C}^I$.

(2) We first establish existence of a partition satisfying (a)–(c), by induction on N. The result is obvious for $N = 1$, so assume the result holds for all integers up to and including some $N \geq 1$, and let $A \in \mathcal{P}_{N+1}(\mathbb{C})$. Let $\{I_1, \dots, I_k\}$ be the partition satisfying properties (a)–(c) for the $N \times N$ upper-left principal submatrix of A, and, for each j, fix a non-negative number, denoted by α_j, which is a G-orbit representative for all entries in the diagonal block A_{I_j}.

Without loss of generality, we may assume $\alpha_j \neq 0$ for all j, since if $a_{mm} = 0$ then $a_{mn} = a_{nm} = 0$ for all n, by positivity. We consider three different cases.

Case 1: $A_{I_j \cup \{N+1\}} \notin \mathcal{P}^1_{|I_j|+1}(\alpha_j \cdot G)$ for all j.

In this case, the partition $\{I_1, \dots, I_k, I_{k+1} := \{N+1\}\}$ clearly has the desired properties.

Case 2: $A_{I_j \cup \{N+1\}} \in \mathcal{P}^1_{|I_j|+1}(\alpha_j \cdot G)$ for a single value of j.

In this case, augmenting I_j with $N+1$ yields the desired partition.

Case 3: $A_{I_i \cup \{N+1\}} \in \mathcal{P}^1_{|I_i|+1}(\alpha_i \cdot G)$ and $A_{I_j \cup \{N+1\}} \in \mathcal{P}^1_{|I_j|+1}(\alpha_j \cdot G)$ for distinct i and j.

We show that this case cannot occur. By the induction hypothesis, there exists $m \in I_i$ and $n \in I_j$ such that the 2×2 minor $A_{\{m,n\}} \notin \mathcal{P}^1_2(\alpha_i \cdot G)$. Now set $a_{N+1,N+1} = \alpha$; by assumption, there exist $g, h \in G$ such that $a_{m,N+1} = \alpha g$ and $a_{n,N+1} = \alpha h$. Thus

$$A_{\{m,n,N+1\}} = \begin{pmatrix} \alpha|g|^2 & a_{mn} & \alpha g \\ \overline{a_{mn}} & \alpha|h|^2 & \alpha h \\ \alpha\overline{g} & \alpha\overline{h} & \alpha \end{pmatrix}, \tag{5.2}$$

so

$$0 \leq \det A_{\{m,n,N+1\}} = -\alpha|\alpha g\overline{h} - a_{mn}|^2$$

and therefore $A_{\{m,n\}} \in \mathcal{P}^1_2(\alpha_i \cdot G)$, which is a contradiction.

This completes the inductive step, and existence follows.

To prove the uniqueness of the decomposition, suppose $\{I_1, \dots, I_k\}$ and $\{J_1, \dots, J_{k'}\}$ are two partitions associated to a matrix $A \in \mathcal{P}_N(\mathbb{C})$, and satisfying the desired properties. Without loss of generality, assume $N \in I_i \cap J_j$, and $I_i \neq J_j$. Let $\alpha := a_{N,N}$ and note that, since I_i and J_j are distinct and maximal, there exist $m \in I_i$ and $n \in J_j$ such that $m \neq n$ and $A_{\{m,n\}} \notin \mathcal{P}^1_2(\alpha_i \cdot G)$ as above. But then the principal minor $A_{\{m,n,N\}}$ is of the form (5.2), which is impossible, as seen previously. It follows that $I_i = J_j$ and the partition is unique, again by induction on N.

(3) If $|c_{ij}| = 1$ for some $i \neq j$, then $A_{I_i \cup I_j} = \mathbf{v}\mathbf{v}^* \in \mathcal{P}^1_{|I_i \cup I_j|}(\alpha \cdot G)$ for some α, where $\mathbf{v} := (\mathbf{u}_i^*, c_{ij}\mathbf{u}_j^*)^*$. This contradicts the maximality of I_i and I_j, so any off-diagonal term $c_{ij} \in D(0,1)$. □

Remark 5.2. It is natural to ask if a similar result to Theorem 5.1 holds if we assume the blocks to have rank bounded above, but not necessarily by 1. This is, however, false, as verified by the example $A = \mathrm{Id}_3$: the partitions $\{\{1,2\},\{3\}\}$, $\{\{2,3\},\{1\}\}$, and $\{\{1,3\},\{2\}\}$ are all such that $A_{I_i \times I_j}$ has rank at most 2. However, A has rank 3, so there is no unique maximal partition when we allow blocks to have rank 2 or higher.

Using the maximal partition corresponding to each subgroup $G \subset \mathbb{C}^\times$, we define a stratification of the cone $\mathcal{P}_N(\mathbb{C})$. The following notation will be useful.

Definition 5.3. Fix a subgroup $G \subset \mathbb{C}^\times$ and an integer $N \geq 1$.

(1) Define $(\Pi_N, \prec)$ to be the partially ordered set of partitions of $\{1,\dots,N\}$, with $\pi' \prec \pi$ if π is a refinement of π'. Given a partition $\pi = \{I_1,\dots,I_k\} \in \Pi_N$, let $|\pi| := k$ denote the size of π.

(2) Given a non-zero matrix $A \in \mathcal{P}_N(\mathbb{C})$, define $\pi^G(A) \in \Pi_N$ to be the unique maximal partition described in Theorem 5.1. Also define $\pi^G(\mathbf{0}_{N\times N})$ to be the indiscrete partition $\{\{1,\dots,N\}\}$.

(3) Given a partition $\pi = \{I_1,\dots,I_k\} \in \Pi_N$, let
$$\mathcal{S}_\pi^G := \{A \in \mathcal{P}_N(\mathbb{C}) : \pi^G(A) = \pi\}. \tag{5.3}$$

Given a subgroup G of $\mathbb{C}^\times$, there is a natural stratification of the cone according to the structure studied in Theorem 5.1:
$$\mathcal{P}_N(\mathbb{C}) = \bigsqcup_{\pi \in \Pi_N} \mathcal{S}_\pi^G,$$
where the set of strata is in bijection with Π_N. The following result discusses basic properties of these Schubert cell-type strata.

Proposition 5.4. *Fix a subgroup $G \subset \mathbb{C}^\times$ and an integer $N \geq 1$.*

(1) *For any partition $\pi \in \Pi_N$, the set $\mathcal{S}_\pi^G$ has real dimension $|\pi|^2 + (N - |\pi|)\dim_{\mathbb{R}} G$, and closure*
$$\overline{\mathcal{S}_\pi^G} = \bigsqcup_{\pi' \prec \pi} \mathcal{S}_{\pi'}^G.$$

(2) *For any $A \in \mathcal{P}_N(\mathbb{C})$, the rank of A is at most $|\pi^{\mathbb{C}^\times}(A)|$.*

Note that $G \subset \mathbb{C}^\times$ can have real dimension $0, 1$, or 2.

Proof.

(1) Suppose $\pi^G(A) = \{I_1,\dots,I_k\}$. Theorem 5.1 implies that a generic matrix in $\mathcal{S}_\pi^G$ is created from a unique matrix $C \in \mathcal{P}_k(\mathbb{C}^\times)$ with only ones on the diagonal, and unique non-zero vectors $\mathbf{u}_j \in \mathbb{C}^{I_j}$ with $\mathbf{u}_{j,1} \in [0,\infty)$ and $\mathbf{u}_{j,l} \in \mathbf{u}_{j,1} \cdot G$ for all $l > 1$. Thus the degrees of freedom for A equal those for the strictly upper-triangular entries of C and for the $\mathbf{u}_j$, i.e.,
$$\begin{aligned}\dim_{\mathbb{R}} \mathcal{S}_\pi^G &= \binom{k}{2}\dim_{\mathbb{R}} \mathbb{C} + \sum_{j=1}^k (1 + (|I_j| - 1)\dim_{\mathbb{R}} G)\\ &= k^2 + (N-k)\dim_{\mathbb{R}} G.\end{aligned}$$
The second observation is straightforward.

(2) This is an immediate consequence of Theorem 5.1(3). $\square$

The following corollary provides a decomposition of a matrix $A \in \mathcal{P}_N(\mathbb{C})$ that will be very useful for studying the kernel of $c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}$.

Corollary 5.5. *Let $A \in \mathcal{P}_N(\mathbb{C})$ and let G be a multiplicative subgroup of S^1. There is a unique partition $\{I_1, \dots, I_k\}$ of $\{1, \dots, N\}$ such that the corresponding diagonal blocks A_{I_j} of A satisfy the following properties.*

1. *The entries in each diagonal block A_{I_j} belong to $\alpha_j \cdot G$ for some $\alpha_j \geq 0$.*
2. *The diagonal blocks have maximal size, i.e., each diagonal block is not contained in a larger diagonal block with entries in $\alpha_j \cdot G$ for some α_j.*

If, moreover, $G = \{1\}$, then A has rank at most k.

Finally, if $\{I_1, \dots, I_k\}$ is any partition satisfying (1) but not necessarily (2), then the entries in every off-diagonal block $A_{I_i \times I_j}$ also share the property that they lie in a single G-orbit in $\mathbb{C}$.

Notice that the result follows from Theorem 5.1 because if $G \subset S^1$, then every block with entries in a single G-orbit automatically has rank at most one, by Theorem 5.8 below.

In what follows, we use Corollary 5.5 with the following two choices of the subgroup G:

1. $G = \{1\}$, in which case all entries in each block of A are equal;
2. $G = S^1$, so all entries in each block of A have equal modulus.

Remark 5.6. Remark that the diagonal blocks of A as in Corollary 5.5 may be 1×1 (for example, in the case where the diagonal entries of A are all distinct). Moreover, the partition of the indices as in Corollary 5.5 does not determine the rank of A. For instance, let $\omega_1, \dots, \omega_N \in S^1$ be pairwise distinct, and let $\mathbf{u} := (\omega_1, \dots, \omega_N)^*$. Then the identity matrix Id_N and $\mathbf{u}\mathbf{u}^*$ have different ranks for $N \geq 2$, but both matrices correspond to the partition of $\{1, \dots, N\}$ into singleton subsets.

5.2. **Simultaneous kernels of Hadamard powers.** We now state and prove the main result of this section, which in particular classifies the simultaneous kernels of Hadamard powers of a positive semidefinite matrix.

Theorem 5.7. *Let $A \in \mathcal{P}_N(\mathbb{C})$ and let $\{I_1, \dots, I_k\}$ be the unique partition of $\{1, \dots, N\}$ satisfying the two conditions of Corollary 5.5 with $G = \{1\}$. Fix $B \in \mathcal{P}_N(\mathbb{C})$ with no zero diagonal entries, and let $c_0, \dots, c_{N-1} > 0$. Then*

$$\ker\bigl(B \circ (c_0 \mathbf{1}_{N \times N} + \cdots + c_{N-1} A^{\circ(N-1)})\bigr) = \bigcap_{n \geq 0} \ker(B \circ A^{\circ n}) = \ker B_{I_1} \oplus \cdots \oplus \ker B_{I_k},$$

where B_{I_j} are the diagonal blocks of B corresponding to the partition $\{I_1, \dots, I_k\}$.

Note that when the Hadamard product is replaced by the standard matrix product, the simultaneous kernel $\cap_{n \geq 1} \ker A^n$ equals $\ker A$. In contrast, characterizing the simultaneous kernels of Hadamard powers is a challenging problem. Also observe in Theorem 5.7 that the simultaneous kernel does not depend on $c_0, \dots, c_{N-1}$.

The proof of Theorem 5.7 repeatedly uses a technical result, which we quote here for convenience.

Theorem 5.8 (Hershkowitz–Neumann–Schneider, [27, Theorem 2.2])**.** *Given an $N \times N$ complex matrix A, where $N \geq 1$, the following are equivalent.*

1. *A is positive semidefinite with entries of modulus 0 or 1, i.e., $A \in \mathcal{P}_N(S^1 \cup \{0\})$.*

(2) *There exist a diagonal matrix D, all of whose diagonal entries lie in S^1, as well as a permutation matrix Q, such that $(QD)^{-1}A(QD)$ is a block diagonal matrix with each diagonal block a square matrix of either all ones or all zeros.*

Equipped with this result, we now prove the above theorem.

Proof of Theorem 5.7. We begin by showing the first equality. One inclusion is immediate; for the reverse inclusion, let $\mathbf{u} \in \ker\big(B \circ (c_0 \mathbf{1}_{N\times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)})\big)$. Then $\mathbf{u} \in \ker(B \circ A^{\circ n})$ for $0 \leq n \leq N-1$, since A and B are positive semidefinite. Applying Lemma 4.3, we conclude that $\mathbf{u} \in \bigcap_{n\geq 0} \ker(B \circ A^{\circ n})$.

We now prove the second equality. Let $\{J_1, \dots, J_l\}$ be a partition of $\{1, \dots, N\}$ as in Corollary 5.5 with $G = S^1$, i.e., with the entries in the diagonal blocks having the same absolute value, instead of necessarily being constant. Clearly, $\{I_1, \dots, I_k\}$ is a refinement of the partition $\{J_1, \dots, J_l\}$. We proceed in three steps. We first show that

$$\bigcap_{n\geq 0} \ker(B \circ A^{\circ n}) \subset \bigcap_{m\geq 1} \ker(B_{J_1} \circ A_{J_1}^{\circ m}) \oplus \cdots \oplus \bigcap_{m\geq 1} \ker(B_{J_l} \circ A_{J_l}^{\circ m}). \tag{5.4}$$

We then prove that each kernel of the form $\bigcap_{m\geq 1} \ker(B_{J_i} \circ A_{J_i}^{\circ m})$ further decomposes into $\ker B_{I_{i_1}} \oplus \cdots \oplus \ker B_{I_{i_p}}$, where $J_i = I_{i_1} \cup \cdots \cup I_{i_p}$. Finally, we prove the reverse inclusion, i.e.,

$$\bigoplus_{m=1}^{k} \ker B_{I_m} \subset \bigcap_{n\geq 0} \ker(B \circ A^{\circ n}).$$

Equation (5.4) is obvious if $l = 1$, so assume $l \geq 2$. Let i satisfy $a_{ii} \geq \max_{j=1}^N a_{jj} > 0$ and suppose, without loss of generality, that $i \in J_1$. Now write the matrices A and B in block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix},$$

where the $(1,1)$ blocks correspond to the $J_1 \times J_1$ entries, and the $(1,2)$ blocks correspond to the $J_1 \times J_1^c$ entries of the matrices, where $J_1^c := J_2 \cup \cdots \cup J_l$. Then, by Theorem 5.8, we conclude that $a_{ii}^{-1} A_{11} = \mathbf{v}\mathbf{v}^*$ for some $\mathbf{v} \in (S^1)^{|J_1|}$. Moreover, we claim that the entries of $a_{ii}^{-1} A_{12}$ have modulus less than one. Indeed, since the entries of the matrix $a_{ii}^{-1} A$ lie in the closed unit disc $\overline{D}(0,1)$, by a positivity argument, we can choose a sequence of integer powers $n_k' \to \infty$ such that $A_\infty := \lim_{k\to\infty} (a_{ii}^{-1} A)^{\circ n_k'}$ exists entrywise. Note that $A_\infty \in \mathcal{P}_N(\mathbb{C})$, by the Schur product theorem. Now let $m \in J_1^c$ and consider the submatrix $A' := (A_\infty)_{J_1 \cup \{m\}}$, which is positive semidefinite and has entries with modulus 0 or 1. Thus, by Theorem 5.8, $A' = \mathbf{w}\mathbf{w}^*$ for some $\mathbf{w} \in \mathbb{C}^{|J_1|+1}$ with $|w_i| = 0$ or 1. By the maximality of J_1, it follows that $|a_{ii}^{-1} a_{j,m}| < 1$ for all $j \in J_1$ and so all the entries of $a_{ii}^{-1} A_{12}$ have modulus less than 1.

Now let $\mathbf{u} = (\mathbf{u}_1^*, \mathbf{u}_2^*)^* \in \bigcap_{n\geq 0} \ker(B \circ A^{\circ n})$, where $\mathbf{u}_1 \in \mathbb{C}^{|J_1|}$, $\mathbf{u}_2 \in \mathbb{C}^{|J_1^c|}$, and $\|\mathbf{u}\| = 1$. We will prove that $(B_{11} \circ A_{11}^{\circ n})\mathbf{u}_1 = 0$ and $(B_{22} \circ A_{22}^{\circ n})\mathbf{u}_2 = 0$ for all $n \geq 1$. To do so, fix $\epsilon \in (0,1)$ and let $\alpha_j := \arg(v_j)/(2\pi)$ for $j = 1, \dots, |J_1|$. Suppose $\{\theta_0 := 1, \theta_1, \dots, \theta_p\}$ is a $\mathbb{Q}$-linearly independent basis of the $\mathbb{Q}$-linear span of $\{1, \alpha_1, \dots, \alpha_{|J_1|}\}$.

Then there exist integers m_{jk}, and an integer $M \geq 1$, such that

$$\alpha_j = \frac{1}{M} \sum_{k=0}^{p} m_{jk} \theta_k \qquad (j = 1, \ldots, |J_1|).$$

By Kronecker's theorem [22, Chapter 23], since θ_k/M are $\mathbb{Q}$-linearly independent, there exist sequences of integers $(n_r)_{r=1}^\infty$ and $(p_r)_{r=1}^\infty$ such that $n_r \to \infty$ and $|n_r(\theta_k/M) - (\theta_k/M) - p_r| < \epsilon$ for $0 \leq k \leq p$. It follows that, for any $j,\ j' \in J_1$,

$$\begin{aligned}
|(a_{ii}^{-1} A_{11})_{j,j'}^{n_r} - (a_{ii}^{-1} A_{11})_{j,j'}| &= |\exp\bigl(2\pi i n_r(\alpha_j - \alpha_{j'})\bigr) - \exp\bigl(2\pi i(\alpha_j - \alpha_{j'})\bigr)| \\
&\leq \left| 2\pi \sum_{k=0}^{p} (m_{jk} - m_{j'k}) \Bigl(\frac{n_r\theta_k - \theta_k}{M} - p_r \Bigr) \right| \\
&\leq 2\pi \max_{k=0,\ldots,p} |n_r(\theta_k/M) - (\theta_k/M) - p_r| \sum_{k=0}^{p} |m_{jk} - m_{j'k}| \\
&\leq C_{jj'}\epsilon
\end{aligned}$$

for some constant $C_{jj'} \geq 0$ independent of ϵ. Let $C := \max\{1, C_{jj'} : j, j' \in J_1\}$.

Replacing $(n_r)_{r=1}^\infty$ by a subsequence if necessary, we may assume without loss of generality that $(a_{ii}^{-1}A)^{\circ n_r}$ converges entrywise to a limit

$$A_\infty := \begin{pmatrix} A_{\infty,11} & \mathbf{0}_{|J_1| \times |J_1^c|} \\ \mathbf{0}_{|J_1^c| \times |J_1|} & A_{\infty,22} \end{pmatrix}.$$

Passing to a further subsequence, we may also assume the entries of $(a_{ii}^{-1}A_{11})^{\circ n_r} - A_{\infty,11}$ are at most $C\epsilon$ in modulus. Moreover,

$$\sum_{k=1}^{2} \mathbf{u}_k^*(B_{kk} \circ A_{\infty,kk})\mathbf{u}_k = \lim_{r\to\infty} \mathbf{u}^*(B \circ (a_{ii}^{-1}A)^{\circ n_r})\mathbf{u} = 0,$$

whence $(B_{kk} \circ A_{\infty,kk})\mathbf{u}_k = 0$ for $k = 1,\ 2$. Therefore, since $\|\mathbf{u}_1\| \leq 1$,

$$\begin{aligned}
&\|(B_{11} \circ A_{11})\mathbf{u}_1\| \\
&\quad \leq a_{ii}\|\bigl(B_{11} \circ (a_{ii}^{-1}A_{11} - (a_{ii}^{-1}A_{11})^{\circ n_r})\bigr)\mathbf{u}_1\| + \|(B_{11} \circ \bigl((a_{ii}^{-1}A_{11})^{\circ n_r} - A_{\infty,11}\bigr))\mathbf{u}_1\| \\
&\quad \leq a_{ii}\|B_{11} \circ (a_{ii}^{-1}A_{11} - (a_{ii}^{-1}A_{11})^{\circ n_r})\| + \|B_{11} \circ \bigl((a_{ii}^{-1}A_{11})^{\circ n_r} - A_{\infty,11}\bigr)\| \\
&\quad \leq \max_{j,k \in J_1} |B_{jk}|\,(a_{ii}+1)\,|J_1|\,C\epsilon;
\end{aligned}$$

as ϵ is arbitrary, we must have $(B_{11} \circ A_{11})\mathbf{u}_1 = 0$. Furthermore, since $\mathbf{u} \in \ker B \circ A$, so

$$(B_{11} \circ A_{11})\mathbf{u}_1 + (B_{12} \circ A_{12})\mathbf{u}_2 = 0,$$

whence $(B_{12} \circ A_{12})\mathbf{u}_2 = 0$, and

$$(B_{12} \circ A_{12})^*\mathbf{u}_1 + (B_{22} \circ A_{22})\mathbf{u}_2 = (B_{21} \circ A_{21})\mathbf{u}_1 + (B_{22} \circ A_{22})\mathbf{u}_2 = 0.$$

This implies that

$$0 = \bigl((B_{12} \circ A_{12})\mathbf{u}_2\bigr)^*\mathbf{u}_1 + \mathbf{u}_2^*(B_{22} \circ A_{22})\mathbf{u}_2 = \mathbf{u}_2^*(B_{22} \circ A_{22})\mathbf{u}_2,$$

and therefore $(B_{22} \circ A_{22})\mathbf{u}_2 = 0$, since $B_{22} \circ A_{22}$ is positive semidefinite.

Repeating the same argument, with A replaced by $A^{\circ m}$ for some fixed $m \geq 1$, we conclude that

$$(B_{kk} \circ A_{kk}^{\circ m})\mathbf{u}_k = 0 \qquad (k = 1, 2,\ m \geq 1).$$

Hence

$$\bigcap_{m \geq 1} \ker(B \circ A^{\circ m}) \subset \bigcap_{m \geq 1} \ker(B_{11} \circ A_{11}^{\circ m}) \oplus \bigcap_{m \geq 1} \ker(B_{22} \circ A_{22}^{\circ m}),$$

and we conclude by induction that

$$\bigcap_{n \geq 0} \ker(B \circ A^{\circ n}) \subset \bigcap_{m \geq 1} \ker(B_{J_1} \circ A_{J_1}^{\circ m}) \oplus \cdots \oplus \bigcap_{m \geq 1} \ker(B_{J_l} \circ A_{J_l}^{\circ m}).$$

We now examine the simultaneous kernel of Hadamard powers of a non-zero diagonal block $B_{J_i} \circ A_{J_i}^{\circ m}$. Assume without loss of generality that $|a_{jk}| = 1$ for all j, $k \in J_i$, and that $J_i = I_1 \cup \cdots \cup I_t$ for some integer $t \geq 1$. By Theorem 5.8, we obtain $A_{J_i} = \mathbf{v}\mathbf{v}^*$ for some $\mathbf{v} \in (S^1)^{|J_i|}$. It follows that A_{J_i} is itself a block matrix with 1 throughout each diagonal sub-block. Thus we can write

$$\mathbf{v} = (\lambda_1 \mathbf{1}_{1 \times n_1}, \ldots, \lambda_t \mathbf{1}_{1 \times n_t})^T \quad \text{and} \quad A_{J_i} = \mathbf{v}\mathbf{v}^* = (\lambda_i \overline{\lambda_j} \mathbf{1}_{n_i \times n_j})_{i,j=1}^t,$$

with $\lambda_1, \ldots, \lambda_t \in S^1$ pairwise distinct, and $|J_1| = n_1 + \cdots + n_t$.

Now let $\mathbf{u} \in \bigcap_{m \geq 1} \ker(B_{J_i} \circ A_{J_i}^{\circ m})$ and let $\mathbf{u} = (\mathbf{u}_1^*, \ldots, \mathbf{u}_t^*)^* \in \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_t}$ be the decomposition of $\mathbf{u}$ corresponding to the partition $\{I_1, \ldots, I_t\}$ of J_i. Let $B_{jk} := B_{I_j, I_k}$; we claim that $\mathbf{u}_j \in \ker B_{jj}$ for all j. Note that

$$\sum_{k=1}^t B_{jk} (\lambda_j \overline{\lambda_k})^m \mathbf{u}_k = 0 \quad \forall m \geq 1,$$

from which it follows that

$$\sum_{k=1}^t (\mathbf{u}_j^* B_{jk} \mathbf{u}_k) \big(\overline{\lambda_k}\big)^{m-1} = 0 \quad \forall m \geq 1.$$

Thus, for fixed j, the vector $(\mathbf{u}_j^* B_{jk} \mathbf{u}_k)_{k=1}^t$ belongs to the kernel of the transpose of the Vandermonde matrix

$$V = (\overline{\lambda_j}^{k-1})_{j,k=1}^t.$$

Since the λ_j are distinct and non-zero, the matrix V is non-singular. Consequently, $\mathbf{u}_j^* B_{jk} \mathbf{u}_k = 0$ for all j, k; in particular, $\mathbf{u}_j^* B_{jj} \mathbf{u}_j = 0$, and therefore $\mathbf{u}_j \in \ker B_{jj}$. This completes this step and shows that

$$\bigcap_{n \geq 0} \ker(B \circ A^{\circ n}) \subset \ker B_{I_1} \oplus \cdots \oplus \ker B_{I_t}.$$

We now prove the reverse inclusion. First, we claim that if $C = (C_{ij})_{i,j=1}^t$ is a block matrix in $\mathcal{P}_N(\mathbb{C})$, and $\mathbf{u} = (\mathbf{u}_1^*, \ldots, \mathbf{u}_t^*)^* \in \mathbb{C}^N$ is the corresponding block decomposition of $\mathbf{u}$, then

$$\mathbf{u} \in \ker C_{11} \oplus \cdots \oplus \ker C_{tt} \quad \Longrightarrow \quad \mathbf{u} \in \ker C. \tag{5.5}$$

We prove the claim by induction on t. The base case of $t = 1$ is obvious. Now suppose (5.5) holds for $t-1$ blocks. Let $\mathbf{u} = (\mathbf{u}_1^*, \dots, \mathbf{u}_t^*)^* = (\mathbf{u}'^*, \mathbf{u}_t^*)^*$, with $\mathbf{u}_j \in \ker C_{jj}$. Then $\mathbf{u}_j^* C_{jj} \mathbf{u}_j = 0$ for all j. Partition C with respect to the same decomposition:

$$C = \begin{pmatrix} C' & C_t' \\ (C_t')^* & C_{tt} \end{pmatrix}.$$

By the induction hypothesis, $\mathbf{u}'^* C' \mathbf{u}' = 0$. On the other hand, for any $\lambda \in \mathbb{R}$,

$$0 \leq (\mathbf{u}'^*, \lambda \mathbf{u}_t^*) C (\mathbf{u}'^*, \lambda \mathbf{u}_t^*)^* = 2\lambda \operatorname{Re}(\mathbf{u}'^* C_t' \mathbf{u}_t).$$

This implies that $\operatorname{Re}(\mathbf{u}'^* C_t' \mathbf{u}_t) = 0$, from which it follows immediately that $\mathbf{u}^* C \mathbf{u} = 0$, and so $\mathbf{u} \in \ker C$. This proves the claim.

Now, to conclude the proof, let $\mathbf{u} \in \ker B_{I_1} \oplus \cdots \oplus \ker B_{I_k}$. Then, for all $n \geq 0$, $\mathbf{u} \in \ker(B_{I_1} \circ A_{I_1}^{\circ n}) \oplus \cdots \oplus \ker(B_{I_k} \circ A_{I_k}^{\circ n})$, since the entries of A are constant in each diagonal block A_{I_j}. It follows by (5.5) that $\mathbf{u} \in \ker(B \circ A^{\circ n})$ for all $n \geq 0$.

This concludes the proof of the theorem. □

Remark 5.9. We note the following consequence of Theorem 5.7. Given a matrix $B \in \mathcal{P}_N(\mathbb{C})$ with no zero diagonal entries, as A runs over the uncountable set $\mathcal{P}_N(\mathbb{C})$, the set of simultaneous kernels

$$\{\cap_{n \geq 0} \ker(B \circ A^{\circ n}) : A \in \mathcal{P}_N(\mathbb{C})\}$$

is, nevertheless, a finite set of subspaces of $\mathbb{C}^N$. Moreover, this finite set is indexed by partitions of the set $\{1, \dots, N\}$. The case when $B = \mathbf{1}_{N \times N}$, or, more generally, when B has no zero diagonal entries, is once again in contrast with the behaviour for the usual matrix powers, and provides a stratification of the cone $\mathcal{P}_N(\mathbb{C})$.

We conclude this section by strengthening Theorem 5.7. Given an integer $N \geq 1$, matrices A, $B \in \mathcal{P}_N(\mathbb{C})$, and a partition $\pi = \{I_1, \dots, I_k\} \in \Pi_N$, let

$$\mathcal{K}_\pi(A, B) := \bigoplus_{j=1}^{k} \bigcap_{n \geq 0} \ker(B_{I_j} \circ A_{I_j}^{\circ n}). \tag{5.6}$$

We have from the proof of Theorem 5.7 that

$$\mathcal{K}_{\{\{1,\dots,N\}\}}(A, B) = \mathcal{K}_{\pi^{S^1}(A)}(A, B) = \mathcal{K}_{\pi^{\{1\}}(A)}(A, B). \tag{5.7}$$

Our final result analyzes the set of partitions π for which Equation (5.7) holds.

Theorem 5.10. *Fix an integer $N \geq 1$ and matrices A, $B \in \mathcal{P}_N(\mathbb{C})$, with B having non-zero diagonal entries. Then,*

$$\{\pi \in \Pi_N : \mathcal{K}_{\{\{1,\dots,N\}\}}(A, B) = \mathcal{K}_\pi(A, B)\} \supset \{\pi : \pi \prec \pi^{\{1\}}(A)\}. \tag{5.8}$$

The reverse inclusion holds if $B \in \mathcal{P}_N^1(\mathbb{C})$.

In particular, for any subgroup $G \subset \mathbb{C}^\times$, Equation (5.7) holds with $\pi^{S^1}(A)$ replaced by $\pi^G(A)$.

Proof. Suppose $\pi^{\{1\}}(A) = \{I_1, \dots, I_k\}$ is a refinement of $\pi = \{J_1, \dots, J_l\}$. Then, restricting to each $J_p \times J_p$ diagonal block,

$$\mathcal{K}_{\pi|_{J_p}}(A_{J_p}, B_{J_p}) := \bigcap_{n \geq 0} \ker(B_{J_p} \circ A_{J_p}^{\circ n})$$

equals $\mathcal{K}_{\pi^{\{1\}}(A_{J_p})}(A_{J_p}, B_{J_p})$, by Theorem 5.7. We are now done by taking the direct sum of the previous equation over all p, since $\bigsqcup_{p=1}^{l} \pi^{\{1\}}(A_{J_p}) = \pi^{\{1\}}(A)$.

Conversely, suppose that $B \in \mathcal{P}_N^1(\mathbb{C})$ and that $\pi^{\{1\}}(A)$ is not a refinement of π. Then, without loss of generality, there exist indices i_1 and i_2 which lie in distinct parts of π but the same part of $\pi^{\{1\}}(A)$. The vector with i_1th entry equal to b_{i_1,i_2}, its i_2th entry equal to $-b_{i_1,i_1}$ and all other entries equal to 0, lies in $\mathcal{K}_{\pi^{\{1\}}(A)}(A, B)$, but it does not lie in $\mathcal{K}_\pi(A, B)$. □

6. Conclusion and survey of known results

It is the aim of the present section to discuss, from a unifying point of view, a collection of old and new computations of sharp bounds for extreme critical values of certain matrix pencils. As mentioned in the introduction, this was a recurrent theme, motivated by theoretical and very applied problems, spanning more than half a century.

In all the examples which follow, we identify a numerical evaluation of the extreme critical value of a concrete matrix pencil. An authoritative source for the spectral theory of polynomial pencils of matrices is [35].

Specifically, Equation (6.1) below provides an accessible, often computationally effective, way of expressing the rather elusive $\mathcal{C}(h; g; \mathcal{P})$, which is, by definition, the smallest real constant C satisfying

$$g[A] \leq C h[A], \qquad \text{for all matrices } A \in \mathcal{P}.$$

We do not exclude above the case $C = \infty$, which means that no uniform bound between $g[A]$ and $h[A]$ exists. A second general observation is the stability of the bound as a function of the matrix set: more precisely, quite a few examples below share the property

$$\mathcal{C}(h; g; \mathcal{P}) = \mathcal{C}(h; g; \mathcal{P}'),$$

where $\mathcal{P}' \subset \mathcal{P}$ is a much smaller class of matrices.

(1) The first of Schoenberg's celebrated theorems proved in [42] involves convergent series in Gegenbauer polynomials. The result can be formulated as a matrix pencil critical-value problem, as follows: fix an integer $d \geq 2$, set $K := [-1, 1]$, and define

$$h(z) = \sum_{n \geq 0} h_n C_n^{(\lambda)}(z) \qquad \text{and} \qquad g(z) = \sum_{n \geq 0} g_n C_n^{(\lambda)}(z),$$

where $h_n \in [0, \infty)$, $g_n \in \mathbb{R}$, $\lambda = (d-2)/2$, and $C_n^{(\lambda)}$ is the corresponding Gegenbauer or Chebyshev polynomial. Also let $\mathcal{P}$ denote the set of all correlation matrices with rank at most d but of arbitrary dimension. Then Schoenberg's

Theorem 2.1(1) asserts that

$$\mathcal{C}(h;g;\mathcal{P}) = \sup_{n:g_n>0} \frac{g_n}{h_n}. \tag{6.1}$$

When $h_n = 0$ and $g_n > 0$ for some index n, the constant $\mathcal{C}(h;g;\mathcal{P})$ is equal to infinity, whence there is no uniform bound for $g[A]$ in terms of $h[A]$, when taken over all matrices $A \in \mathcal{P}$.

(2) Schoenberg's second landmark result from [42], as well as its subsequent extensions by Christensen and Ressel, Hiai, and others (see the references after Theorem 2.1), can also be rephrased as an extreme critical-value problem, as follows. Set $K := (-\rho, \rho)$, where $0 < \rho \leq \infty$, and consider the convergent power series

$$h(z) = \sum_{n\geq 0} h_n z^n, \; g(z) = \sum_{n \geq 0} g_n z^n : (-\rho,\rho) \to \mathbb{R}, \tag{6.2}$$

where $h_n \in [0,\infty)$ and $g_n \in \mathbb{R}$ for all n, and $\mathcal{P} = \bigcup_{N\geq 1} \mathcal{P}_N((-\rho,\rho))$. Then $\mathcal{C}(h;g;\mathcal{P})$ can be computed as in Equation (6.1).

(3) In the papers [31, 41], Rudin, working with Kahane, proved that preserving positivity on low-rank Toeplitz matrices already implies absolute monotonicity. In this case, set $K := (-1,1)$, and let $h(z)$ and $g(z)$ to be as in Equation (6.2) with $\rho = 1$. Also let $\mathcal{P} := \bigcup_{N\geq 1} \mathcal{P}_N((-\rho,\rho))$ and let $\mathcal{P}'$ denote the set of Toeplitz matrices of all dimensions and of rank at most 3. Rudin showed in [41, Theorem IV] that it suffices to test the pencil bound on the set $\mathcal{P}_\alpha := \{M(a,b,n,\alpha) : 0 \leq a, b, \; a+b < 1, n \geq 1\}$ for any irrational multiple α of π, where $M(a,b,n,\alpha) \in \mathcal{P}_n^3([-1,1])$ is the Toeplitz matrix with (j,k)th entry $a + b\cos((j-k)\alpha)$.

Then $\mathcal{C}(h;g;\mathcal{P}) = \mathcal{C}(h;g;\mathcal{P}') = \mathcal{C}(h;g;\mathcal{P}_\alpha)$ and Equation (6.1) holds.

(4) A necessary condition for preserving positivity in fixed dimension was provided by Horn in [30]. The condition was subsequently extended by Guillot–Khare–Rajaratnam [18] and is stated in Theorem 2.2 above. This yields a special case of the extreme critical-value problem, with

$$h(z) = \sum_{n=0}^{N-1} h_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{N-1} g_n z^n, \tag{6.3}$$

where $h_n, \; g_n \in [0,\infty)$ for all n, and $\mathcal{P} = \mathcal{P}_N^2((0,\rho))$. Then $\mathcal{C}(h;g;\mathcal{P}) = \mathcal{C}(h;g;\mathcal{P}_N((0,\rho)))$ and Equation (6.1) holds.

(5) The problem of preserving positivity has recently attracted renewed attention, due to its application in the regularization of ultra high-dimensional covariance matrices. The next few observations are along those lines. First, in [18, Proposition 3.17(3)], the authors consider a more general situation than the previous instance, where one replaces the polynomials h, g of degree at most $N-1$ by one of the following.

- A linear combination of N fractional powers z^α, with $K = (0,\rho)$ and $\mathcal{P} = \mathcal{P}_N^1(K)$; here α can be negative.
- A linear combination of N fractional powers z^α and the constant function 1, with $K = [0,\rho)$ and $\mathcal{P} = \mathcal{P}_N^1(K)$; here $0^\alpha := 0$ for $\alpha \in \mathbb{R}$.

- A linear combination of N fractional powers of the form $\phi_\alpha(z) := |z|^\alpha$ or $\psi_\alpha(z) := \operatorname{sgn}(z)|z|^\alpha$, with $K = (-\rho, \rho)$ and $\mathcal{P} = \mathcal{P}_N^1(K)$; here $\phi_\alpha(0) = \psi_\alpha(0) := 0$.

In each case, if h_n, $g_n \in [0, \infty)$ for all n, with both h and g involving the same set of fractional powers, then $\mathcal{C}(h; g; \mathcal{P}) = \mathcal{C}(h; g; \mathcal{P}_N(K))$ and Equation (6.1) holds.

(6) Note that Lemma 2.4 also yields an extreme critical value that can be deduced from either of the two previous cases. In particular, if h and g are power series, as in Equation (6.2), and we define $\mathcal{P} = \bigcup_{N \geq 1} \mathcal{P}_N((0, \rho))$ and $\mathcal{P}' = \bigcup_{N \geq 1} \mathcal{P}_N^1((0, \rho))$, then $\mathcal{C}(h; g; \mathcal{P}) = \mathcal{C}(h; g; \mathcal{P}')$ and Equation (6.1) holds. This result was also shown, using alternate approaches, in [18].

(7) Another application involves entrywise functions preserving positivity on matrices with zero structure according to a tree graph T. Recall that for a graph G with vertex set $\{1, \ldots, N\}$, and a subset $K \subset \mathbb{C}$, the cone $\mathcal{P}_G(K)$ is defined to be the set of matrices $A = (a_{ij}) \in \mathcal{P}_N(K)$ such that if $i \neq j$ and (i, j) is not an edge in G, then $a_{ij} = 0$. An extreme critical-value phenomenon was shown in [19]: fix powers

$$0 < r' < r < s < s' < \infty, \quad r > 1,$$

as well as a measurable set $B \subset (r, s)$. Also fix scalars $a_{r'}$, a_r, a_s, $a_{s'} > 0$ and a measurable function $l : B \to \mathbb{R}$. Now define

$$h(z) = a_{r'} z^{r'} + a_r z^r + a_s z^s + a_{s'} z^{s'}, \qquad g(z) = \int_B l(z) \, \mathrm{d}z \qquad (z \in \mathbb{R}),$$

and $\mathcal{P} = \bigcup_{T \in \mathcal{S}} \mathcal{P}_T((0, \infty))$, where $\mathcal{S}$ is a non-empty set of connected trees on at least 3 vertices. Then [19, Theorem 4.6] shows the existence of a finite threshold: $\mathcal{C}(h; g; \mathcal{P}) \in (0, \infty)$. Note that this is an existence result, in contrast to the sharp bounds obtained in all previous examples.

(8) We turn now to the present paper, in which extreme critical values were obtained for various families of linear pencils. For all of the remaining examples, fix the following notation: $N \geq 1$ and $M \geq 0$ are integers, $\rho \in (0, \infty)$, and

$$\begin{aligned} c_0, c_1, \ldots, c_{N-1} &\in (0, \infty), \\ h_{\mathbf{c}}(z) &= c_0 + \cdots + c_{N-1} z^{N-1}, \\ \text{and} \quad \mathcal{P}_N^1((0, \rho)) &\subset \mathcal{P} \subset \mathcal{P}_N(\overline{D}(0, \rho)). \end{aligned} \tag{6.4}$$

Then the main result of the present paper, Theorem 1.1, says that

$$\mathcal{C}(h_{\mathbf{c}}; z^M; \mathcal{P}) = \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j}.$$

Note that if $M = N$, $\mathcal{P} = \mathcal{P}_N([-\rho, \rho])$, and N tends to infinity, then the extreme critical value grows without bound, thereby recovering Schoenberg's result discussed in the second example above.

(9) The $\mathbb{R}_+$-subadditivity of $\mathcal{C}(h; g; \mathcal{P})$ in its second argument, as in Equation (3.19), immediately yields, for a polynomial $g(z) = \sum_{n=0}^{M} g_n z^n$, that

$$\mathcal{C}(h_{\mathbf{c}}; g; \mathcal{P}) \leq \sum_{n: g_n > 0} g_n \mathcal{C}(h_{\mathbf{c}}; z^n; \mathcal{P}),$$

where the notation is as in (6.4).

(10) When $g(z) = \sum_{M=N}^{\infty} c_M z^M$ is analytic on $D(0, \rho)$ and continuous on $\overline{D(0, \rho)}$, with real coefficients, Theorem 1.3 yields a bound on the corresponding extreme critical value:

$$\mathcal{C}(\mathbf{c}; g; N, \rho) \leq \frac{g_2^{(2N-2)}(\sqrt{\rho})}{2^{N-1}(N-1)!^2} \sum_{j=0}^{N-1} \binom{N-1}{j}^2 \frac{\rho^{N-j-1}}{c_j},$$

where $g_+(z) = \sum_{M \geq N: c_M > 0} c_M z^M$ and $g_2(z) = g_+(z^2)$.

(11) In the case of 2×2 matrices, fix non-negative integers $m < n < p$, set $h_2(z) = c_m z^m + c_n z^n$, and suppose $\mathcal{P}_2^1([0, 1]) \subset \mathcal{P}' \subset \mathcal{P}_2([0, 1])$. Then Theorem 3.10 shows that

$$\mathcal{C}(h_2; z^p; \mathcal{P}') = \frac{c_m c_n (n - m)^2}{c_m (p - m)^2 + c_n (p - n)^2}.$$

(12) The next instance involves Rayleigh quotients for Hadamard powers. In this case, we let $\mathcal{P}$ be the set containing a single non-zero matrix $A \in \mathcal{P}_N(\mathbb{C})$. Define

$$\mathcal{K}(A) = \ker h_{\mathbf{c}}[A] = \ker(c_0 \mathbf{1}_{N \times N} + c_1 A + \cdots + c_{N-1} A^{\circ(N-1)}).$$

Then Propositions 4.1 and 4.2 show that, with notation as in (6.4),

$$\mathcal{C}(h_{\mathbf{c}}; z^M; A) = \max_{\mathbf{v} \in S^{2N-1} \cap \mathcal{K}(A)^\perp} \frac{\mathbf{v}^* A^{\circ M} \mathbf{v}}{\mathbf{v}^* h_{\mathbf{c}}[A] \mathbf{v}} = \varrho(h_{\mathbf{c}}[A]^{\dagger/2} A^{\circ M} h_{\mathbf{c}}[A]^{\dagger/2}) < \infty.$$

In particular, $\mathcal{C}(h_{\mathbf{c}}; z^M; \mathbf{u}\mathbf{u}^*) = (\mathbf{u}^{\circ M})^* h_{\mathbf{c}}[\mathbf{u}\mathbf{u}^*]^\dagger \mathbf{u}^{\circ M}$ for all non-zero $\mathbf{u} \in \mathbb{C}^N$, by Corollary 4.5.

(13) Our final example involves an application of Theorem 5.7, in which we obtained a stratification of the cone $\mathcal{P}_N(\mathbb{C})$ by the set Π_N of partitions of $\{1, \dots, N\}$. Given a partition $\pi \in \Pi_N$, define the stratum $\mathcal{S}_\pi^{\{1\}}$ as in Equation (5.3). The key observation is that the simultaneous kernel map $A \mapsto \mathcal{K}(A)$ is constant on each stratum. In other words, the map

$$\mathcal{K} : \mathcal{P}_N(\mathbb{C}) \longrightarrow \Pi_N \longrightarrow \bigsqcup_{r=0}^{N-1} \mathrm{Gr}(r, \mathbb{C}^N)$$

sends every matrix $A \in \mathcal{S}_\pi^{\{1\}}$ to a fixed subspace

$$\mathcal{K}_\pi := \ker \sum_{j=1}^{|\pi|} \mathbf{1}_{I_j \times I_j} \in \mathrm{Gr}(N - |\pi|, \mathbb{C}^N),$$

where the $N \times N$ matrix $\mathbf{1}_{E \times F}$ has (i, j)th entry equal to 1 if $(i, j) \in E \times F$ and equal to 0 otherwise, $|\pi| = k$ denotes the number of parts in the partition $\pi =$

$\{I_1, \dots, I_k\}$, and $\mathrm{Gr}(r, \mathbb{C}^N)$ denotes the complex Grassmann manifold of r-dimensional subspaces of $\mathbb{C}^N$.

Now suppose $h(z)$ is as in (6.4),

$$g(z) = \sum_{n=0}^{M} g_n z^n \quad (g_n \in [0,\infty)), \quad \text{and} \quad \mathcal{P} = \bigsqcup_{\pi \in \Pi_N} \mathcal{S}_\pi^0,$$

where $\mathcal{S}_\pi^0 \subset \mathcal{S}_\pi^{\{1\}}$ is a compact subset of the corresponding stratum. Then

$$\begin{aligned} \mathcal{C}(h_{\mathbf{c}}; g; \mathcal{P}) &\leq \sum_{n : g_n > 0} g_n \mathcal{C}(h_{\mathbf{c}}; z^n; \mathcal{P}) \\ &= \sum_{n : g_n > 0} g_n \max_{\pi \in \Pi_N} \max \Big\{ \frac{\mathbf{v}^* A^{\circ n} \mathbf{v}}{\mathbf{v}^* h_{\mathbf{c}}[A] \mathbf{v}} : \mathbf{v} \in S^{2N-1} \cap \mathcal{K}_\pi^\perp, \ A \in \mathcal{S}_\pi^0 \Big\}, \end{aligned} \tag{6.5}$$

and this is a finite number because the inner maximum is taken over a product of compact sets, and the function being optimized is continuous in both variables.

It would be interesting to investigate the jumping locus of the map $A \mapsto \mathcal{C}(h_{\mathbf{c}}; z^M; A)$. In particular, is this map continuous on the stratum $\mathcal{S}_\pi^{\{1\}}$, for each partition $\pi \in \Pi_N$?

More generally, given any matrix $B \in \mathcal{P}_N(\mathbb{C})$ with no zero diagonal entries, Theorem 5.7 provides a stratification $\mathcal{P}_N(\mathbb{C}) = \bigsqcup_{\pi \in \Pi_N} \mathcal{S}_{B,\pi}^{\{1\}}$ in a similar vein to the above; thus $\mathcal{S}_\pi^{\{1\}} = \mathcal{S}_{\mathbf{1}_{N \times N},\pi}^{\{1\}}$ for all $\pi \in \Pi_N$. Once again, the map $\mathcal{K}$ is constant on each stratum, sending $\mathcal{S}_{B,\pi}^{\{1\}}$ to $\mathcal{K}_{B,\pi}$, say. Then, by Proposition 4.2 and Lemma 4.3, the inequality in (6.5) generalizes, for each fixed B, with $h_{\mathbf{c}}[A]$ and $A^{\circ n}$ replaced by $B \circ h_{\mathbf{c}}[A]$ and $B \circ A^{\circ n}$ for all n, respectively. See the preceding section for full details.

List of symbols. A few ad hoc notations were introduced in the text. We list them below for the convenience of the reader.

- $\mathcal{P}_N^k(K)$ is the set of positive semidefinite $N \times N$ matrices with entries in a subset $K \subset \mathbb{C}$ and of rank at most k.
- $\mathcal{P}_N(K) := \mathcal{P}_N^N(K)$.
- $A^{\circ k}$ is the matrix obtained from A by taking the kth power of each entry.
- $\mathbf{1}_{N \times N}$ is the $N \times N$ matrix with each entry equal to 1.
- $f[A]$ is the result of applying f to each entry of the matrix A.
- $\mathcal{C}(h; g; \mathcal{P})$ is the smallest non-negative constant satisfying $g[A] \leq \mathcal{C}(h; g; \mathcal{P}) h[A]$ for all $A \in \mathcal{P}$.
- $\varrho(A)$ is the spectral radius of the matrix A.

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